HW #21 Solutions:

Find the critical points for the following equations

(a) $f(x,y) = xy^2 + x^2$

(b) $f(x,y) = y^3 + 2xy^2 + x^2$

(c) $f(x,y) = e^x \sin y$

(d) $f(x,y) = xy + x^2 + y^2$ *

(a) $f(x,y) = xy^2 + x^2$

$\nabla f(x,y) = \langle y^2 + 2x,\ 2xy \rangle = \langle 0,0 \rangle$

- $f_x(x,y) = y^2 + 2x = 0 \Rightarrow x = \frac{-y^2}{2}$
- $f_y(x,y) = 2xy = 0 \Rightarrow 2xy = -y^3 = 0 \Rightarrow \boxed{y=0} \Rightarrow \boxed{x=0}$

$f_{xx} = 2 \qquad f_{xy} = 2y$

$f_{yy} = 2x$

$$d = (2)(2x) - (2y)^2$$
$$= 4x - 4y^2$$
$$d(0,0) = 4(0) - 4(0)^2 = 0$$

Critical point: $(0,0)$. Classification is inconclusive

(b) $f(x,y) = y^3 + 2xy^2 + x^2$

$\nabla f(x,y) = \langle 2y^2 + 2x,\ 3y^2 + 4xy \rangle = \langle 0,0 \rangle$

- $f_x(x,y) = 2y^2 + 2x = 0 \Rightarrow -y^2 = x$
- $f_y(x,y) = 3y^2 + 4xy = 0 \Rightarrow 3y^2 - 4y^3 = 0 \Rightarrow y^2(3 - 4y) = 0$

$\Rightarrow y = 0$ or $y = \frac{3}{4}$

$\Rightarrow x = 0$ or $x = \frac{-9}{16}$

Critical points: $(0,0), \left(\frac{-9}{16}, \frac{3}{4}\right)$

- $f_{xx}(x,y) = 2$
- $f_{yy}(x,y) = 6y + 4x$
- $f_{xy}(x,y) = 4y$

$$d = (2)(6y+4x) - (4y)^2$$
$$= 12y + 8x - 16y^2$$

$d(0,0) = 0 + 0 - .0 = 0$ so inconclusive

$$d\left(\frac{-9}{16}, \frac{3}{4}\right) = \frac{(\cancel{12})(3)}{\cancel{4}} + \frac{(\cancel{8})(-9)}{(\cancel{16})} - \cancel{16}\left(\frac{9}{\cancel{16}}\right)$$

$$= 9 - 9 - 9 = -9$$

$\left(\frac{-9}{16}, \frac{3}{4}\right)$ is a saddle point.

(c.) $f(x,y) = e^x \sin y$

$\nabla f(x,y) = \langle e^x \sin y, \ e^x \cos(y) \rangle = \langle 0, 0 \rangle$

- $e^x \sin y = 0 \Rightarrow y = \ldots -\pi, 0, \pi, 2\pi, \ldots$
- $e^x \cos y = 0 \Rightarrow y = \ldots \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

So we cannot make this happen, therefore, there is no critical point.

(d) $f(x,y) = xy + x^2 + y^2$

$\nabla f(x,y) = \langle y + 2x, \; x + 2y \rangle$

- $f_x(x,y) = y + 2x = 0 \Rightarrow y = -2x$
- $f_y(x,y) = x + 2y = 0 \Rightarrow x - 4x = 0$

$\Rightarrow -3x = 0$

$x = 0 \Rightarrow y = 0$

Critical point : $(0,0)$

$f_{xx}(x,y) = 2$

$f_{yy}(x,y) = 2$ $\quad d = 4 - 1 = 3 > 0$

$f_{xy}(x,y) = 1$ $\quad f_{xx} = 2 > 0$

So $(0,0)$ is a minimum