Homework 20 solutions:

1.) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ @ $(3,2,6)$

approximate $f(3.02, 1.97, 5.99)$

$f(3,2,6) = \sqrt{49} = 7$

$F(x,y,z,w) = \sqrt{x^2+y^2+z^2} - w$

$$\nabla F = \left\langle \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}}, \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2+z^2}}, \frac{\cancel{2}z}{\cancel{2}\sqrt{x^2+y^2+z^2}}, -1 \right\rangle$$

$$\nabla F(3,2,6) = \left\langle \frac{3}{\sqrt{49}}, \frac{2}{\sqrt{49}}, \frac{6}{\sqrt{49}}, -1 \right\rangle$$

$$= \left\langle \frac{3}{7}, \frac{2}{7}, \frac{6}{7}, -1 \right\rangle$$

pt: $(3,2,6,7)$

"Plane": $\boxed{0 = \frac{3}{7}(x-3) + \frac{2}{7}(y-2) + \frac{6}{7}(z-6) - (w-7)}$

$$\Rightarrow \quad w = 7 + \frac{3}{7}(x-3) + \frac{2}{7}(y-2) + \frac{6}{7}(z-6)$$

Approximation:

$$f(3.02, 1.97, 5.99) \approx 7 + \frac{3}{7}(.02) + \frac{2}{7}(-.03) + \frac{6}{7}(-.01)$$

$$= 7 + \frac{\cancel{.06} - \cancel{.06} - .06}{7}$$

$$= 7 - \frac{6}{700} = \frac{4900-6}{700} = \frac{4894}{700}$$

$$= \boxed{\frac{2447}{350}} \approx 6.99$$

2. $\cdot\ f(1 \text{ billion}, 90) = 200$

$\cdot\ f_x(1 \text{ billion}, 90) = 2000$

$\cdot\ f_x(1 \text{ billion}, 90) = 5$

Estimate $f(2 \text{ billion}, 140)$

$$f(2 \text{ bill}, 140) \approx f(1 \text{ bill}, 90) + f_x(1 \text{ bill}, 90)(x-1) + f_y(1 \text{ bill}, 90)(y-90)$$

$$= 200 + 2000(2-1) + 5(140-90)$$

$$= 200 + 2000 + 250$$

$$= \$2{,}450$$

The estimate is \$2,450. This is <u>not</u> a good estimate because (2, 140) is very far from the point (1, 90).

3. Find Δz if $\Delta x = 0.5$, $\Delta y = 0.7$ and $f(3,1) = 2$, $f_x(3,1) = -1$, $f_y(3,1) = 10$.

$$\Delta z \approx f_x(3,1)\Delta x + f_y(3,1)\Delta y$$

$$= -(0.5) + 10(0.7) = 7 - .5 = \boxed{6.5}$$

4. Find dz/dt.

(a.) $f(x,y) = x^3 + 3x^2y + 3xy^2 + y^3$, $x = 3t$, $y = t^2$.

$f_x = 3x^2 + 6xy + 3y^2$

$f_y = 3x^2 + 6xy + 3y^2$

$\leftrightarrow$ same!

$\frac{dx}{dt} = 3 \quad \frac{dy}{dt} = 2t$

same formula, just factored since $f_x = f_y$

$\frac{dz}{dt} = \left[3(3t)^2 + 6(3t)(t^2) + 3(t^2)^2\right](3 + 2t)$

$= (27t^2 + 18t^3 + 3t^4)(3 + 2t)$

$= 81t^2 + 34t^3 + 9t^4 + 54t^3 + 36t^4 + 6t^5$

$= \boxed{81t^2 + 88t^3 + 45t^4 + 6t^5}$

(10) $f(x,y) = \cos(xy)$, $\quad x = 1/t$, $\quad y = t^3 + t$

$f_x(x,y) = -y\sin(xy) = -(t^3+t)\sin(t^2+1)$

$f_y(x,y) = -x\sin(xy) = -\frac{1}{t}\sin(t^2+1)$

$\frac{dx}{dt} = \frac{-1}{t^2}$, $\quad \frac{dy}{dt} = 3t^2 + 1$

$\frac{dz}{dt} = -(t^3+t)\sin(t^2+1)\left(-\frac{1}{t^2}\right) - \frac{1}{t}\sin(t^2+1)(3t^2+1)$

$= \left(t + \frac{1}{t}\right)\sin(t^2+1) - \left(3t + \frac{1}{t}\right)\sin(t^2+1)$

$= \boxed{-2t\sin(t^2+1)}$

4 c.) $f(x,y) = x^2 + y^2$, $x = \sin(2t)$ $y = \cos(2t)$

$f_x = 2x = 2\sin(2t)$

$f_y = 2y = 2\cos(2t)$

$\frac{dx}{dt} = 2\cos(2t)$ $\frac{dy}{dt} = -2\sin(2t)$

$$\frac{dz}{dt} = 4\cos(2t)\sin(2t) - 4\cos(2t)\sin(2t) = \boxed{0}$$

5.) Use the chain rule to determin $\partial z/\partial t$ and $\partial z/\partial s$.

ⓐ $f(x,y) = \arctan(x-y)$, $x = t^2 + s^2$, $y = 2st$

$f_x(x,y) = \frac{1}{1+(x-y)^2}$, $f_y(x,y) = \frac{-1}{1+(x-y)^2}$

$$x - y = t^2 + s^2 - 2st$$
$$= t^2 - 2st + s^2 = (t-s)^2$$

$\frac{\partial x}{\partial t} = 2t$ $\frac{\partial x}{\partial s} = 2s$ $\frac{\partial y}{\partial t} = 2s$ $\frac{\partial y}{\partial s} = 2t$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= \frac{1}{1+(t-s)^4}(2t) + \frac{-1}{1+(t-s)^4}(2s) = \boxed{\frac{2t-2s}{1+(t-s)^4}}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \frac{1}{1+(t-s)^4}\cdot 2s + \frac{-1}{1+(t-s)^4}\cdot 2t = \boxed{\frac{2s-2t}{1+(t-s)^4}}$$

(b) $f(x,y) = \cos(x)\sin(y)$, $\quad x = s^2t^2$, $\quad y = st$

$$f_x(x,y) = -\sin(x)\sin(y) = -\sin(s^2t^2)\sin(st)$$

$$f_y(x,y) = \cos(x)\cos(y) = \cos(s^2t^2)\cos(st)$$

$$\frac{\partial x}{\partial t} = 2s^2t \qquad \frac{\partial y}{\partial t} = s$$

$$\frac{\partial x}{\partial s} = 2st^2 \qquad \frac{\partial y}{\partial s} = t$$

$$\boxed{\frac{\partial z}{\partial t} = -2s^2t\sin(s^2t^2)\sin(st) + s\cos(s^2t^2)\cos(st)}$$

$$\boxed{\frac{\partial z}{\partial s} = -2st^2\sin(s^2t^2)\sin(st) + t\cos(s^2t^2)\cos(st)}$$