MATH 231: Calculus of Several Variables
Section 1, 107 Ag Sc & Ind Bldg,
TR 9:05 AM - 9:55 AM

Homework 13: Due Thursday, Oct 24

1. Find the (i) unit tangent, (ii) normal, and (iii) binormal vectors for the following functions.
   (a) $\vec{r}(t) = (1+t^2)\vec{i} + (t^3)\vec{j}$
   (b) $\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j} + t\vec{k}$

2. Find a_T for the following functions.
   (a) $\vec{r}(t) = (1+t)\vec{i} + (t^2 - 2t)\vec{j}$
   (b) $\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j} + t\vec{k}$

1. a.) (i.) $\vec{T}(t) = \frac{1}{\sqrt{4+9t^2}} \langle 2, 3t \rangle$

(ii) $\vec{N}(t) = \frac{1}{\sqrt{4+9t^2}} \langle -3t, 2 \rangle$

iii.) $\vec{B}(t) = \langle 0, 0, 1 \rangle$

b.) (i) $\vec{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), \cos(t), 1 \rangle$

(ii) $\vec{N}(t) = \langle -\cos(t), -\sin(t), 0 \rangle$

(iii) $\vec{B}(t) = \frac{1}{\sqrt{2}} \langle \sin(t), -\cos(t), 1 \rangle$

2. a.) $a_T = \frac{4t-4}{\sqrt{4t^2-8t+5}}$

b.) $a_T = 0$

Hw #13 Solutions:

1. a.) $\vec{r}(t) = (1+t^2)\vec{i} + (t^3-2)\vec{j}$

First, let's rewrite this with the bracket notation

$\vec{r}(t) = \langle 1+t^2, t^3-2 \rangle$

i.) $\vec{r}\,'(t) = \langle 2t, 3t^2 \rangle$

$|\vec{r}\,'(t)| = \sqrt{4t^2 + 9t^4}$

$= t\sqrt{4+9t^2}$

$$\boxed{\vec{T}(t) = \left\langle \frac{2}{\sqrt{4+9t^2}}, \frac{3t}{\sqrt{4+9t^2}} \right\rangle}$$

ii.) $\vec{T}\,'(t) = \left\langle \frac{-18t}{(4+9t^2)^{3/2}}, \frac{12}{(4+9t^2)^{3/2}} \right\rangle$

$$= \frac{6}{(4+9t^2)^{3/2}} \langle -3t, 2 \rangle$$

$$|\vec{T}\,'(t)| = \left| \frac{6}{(4+9t^2)^{3/2}} \right| \sqrt{9t^2+4}$$

$$= \frac{6}{(4+9t^2)}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{6/(4+9t^2)^{3/2}\langle -3t, 2\rangle}{6/(4+9t^2)}$$

$$\boxed{\vec{N}(t) = \frac{1}{\sqrt{4+9t^2}}\langle -3t, 2\rangle}$$

iii.) $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

$$= \underbrace{\left(\frac{1}{\sqrt{4+9t^2}}\right)}_{\text{from } \vec{T}(t)}\underbrace{\left(\frac{1}{\sqrt{4+9t^2}}\right)}_{\text{from } \vec{N}(t)}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3t & 0 \\ -3t & 2 & 0 \end{vmatrix}$$

$$= \frac{1}{(4+9t^2)}\left[0\vec{i} - 0\vec{j} + (4+9t^2)\vec{k}\right]$$

$$= \boxed{\langle 0, 0, 1\rangle}$$

b.) $\vec{r}(t) = \langle \cos(t), \sin(t), t\rangle$

(i) $\vec{r}'(t) = \langle -\sin(t), \cos(t), 1\rangle$

$|\vec{r}'(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$

$$\boxed{\vec{T}(t) = \frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1\rangle}$$

(ii.) $\vec{T}'(t) = \frac{1}{\sqrt{2}}\langle -\cos(t), -\sin(t), 0\rangle$

$$|\vec{T}'(t)| = \frac{1}{\sqrt{2}}\sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{2}}$$

$$\boxed{\vec{N}(t) = \langle -\cos(t), -\sin(t), 0\rangle}$$

(iii.) $\vec{B} = \vec{T}(t) \times \vec{N}(t)$

$$= \frac{1}{\sqrt{2}}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(t) & \cos(t) & 1 \\ -\cos(t) & -\sin(t) & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}}\left[(\sin(t))\vec{i} - (\cos(t))\vec{j} + (\sin^2(t) + \cos^2(t))\vec{k}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\sin(t)\,\vec{i} - \cos(t)\,\vec{j} + \vec{k}\right]$$

$$= \boxed{\frac{1}{\sqrt{2}}\langle \sin(t), -\cos(t), 1\rangle}$$

2. $a_T = \dfrac{\vec{v}(t) \cdot \vec{a}(t)}{|\vec{v}(t)|}$

a.) $\vec{r}(t) = \langle 1+t,\ t^2-2t \rangle$

$\vec{v}(t) = \vec{r}\,'(t) = \langle 1,\ 2t-2 \rangle$

$|\vec{v}(t)| = \sqrt{1+(2t-2)^2} = \sqrt{1+4t^2-8t+4} = \sqrt{4t^2-8t+5}$

$\vec{a}(t) = \vec{v}\,'(t) = \langle 0,\ 2 \rangle$

$$\boxed{a_T = \frac{4t-4}{\sqrt{4t^2-8t+5}}}$$

b.) $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$

$\vec{v}(t) = \vec{r}\,'(t) = \langle -\sin(t),\ \cos(t),\ 1 \rangle$

$|\vec{v}(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}$

$\vec{a}(t) = \vec{v}\,'(t) = \langle -\cos(t),\ -\sin(t),\ 0 \rangle$

$$a_T = \frac{\sin(t)\cos(t) - \sin(t)\cos(t) + 0}{\sqrt{2}} = \boxed{0}$$