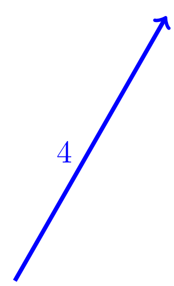
3.2 Vectors

Objectives

- ☐ I can describe and define a vector. I can define magnitude and direction.
- ☐ I understand the difference between a point and a vector.
- ☐ I know how to find the vector that represents displacement from one point to another.
- ☐ I understand why we use the pythagorean theorem to determine the magnitude of a vector.
- ☐ I can add two vectors and I can multiply a scalar by a vector.
- ☐ I know how to calculate the unit vector.

"The suspect is traveling NE at 4 miles per hour."

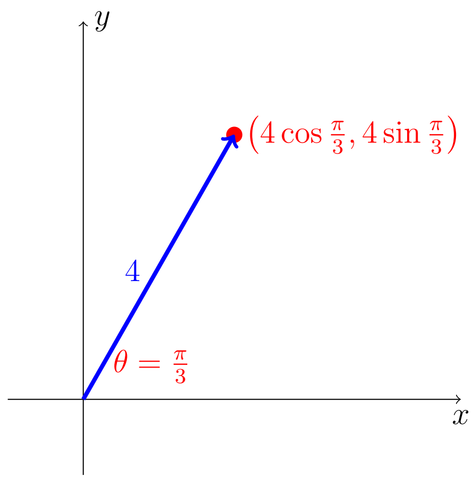
When we're talking about 3-dimensional space, we're interested in statements like the one you see above. It's a statement of **direction**, NE, and of **speed**, 4 miles per hour. How can we represent something like this? We use arrows. The direction it points is the direction. Its length represents speed.



In the picture above, the arrow represents going at a speed of 4 in the direction it's pointing. Observe that this is not something that has a location attached to it. Instead it is a description of movement.

How do we express this object mathematically?

Let's embed it into the plane. Location doesn't matter, so we place it at the origin. Now focus on the point at which the arrow ends. We can describe it based on the angle between the arrow and the x-axis.



We can describe our arrow with the point

$$\left(4\cos\frac{\pi}{3}, 4\sin\frac{\pi}{3}\right).$$

Here, we're using the construction of the unit circle. Remember that the unit circle has radius 1, but the length of the arrow is 4, so we multiply the entries by 4.

The most important thing to remember in this construction is *a point describes a location while an arrow describes movement.* We distinguish between points and arrows by using angle brackets. The arrow is described as

$$\left\langle 4\cos\frac{\pi}{3}, 4\sin\frac{\pi}{3}\right\rangle = \left\langle 2, 2\sqrt{3}\right\rangle.$$

Let's summarize. The arrows *do not have a location.* They point to a **direction** with a speed. Mathematicians call speed a **magnitude** and that's how we will refer to it from now on. Instead of arrows, mathematicians call these objects **vectors**.

Definition 3.2.1 *A* **vector** *is an object with direction and magnitude. It has no location.*

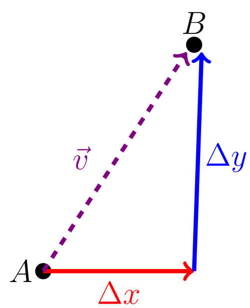
3.2.1 Examples

Example 3.2.1.1 *Find the (two-dimensional) vector that is* $45°$ *above the* x*-axis with length* 7*.*

By our above example, we can see that

$$\vec{v} = \langle 7\cos 45°, 7\sin 45°\rangle = \left\langle \frac{7}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right\rangle$$

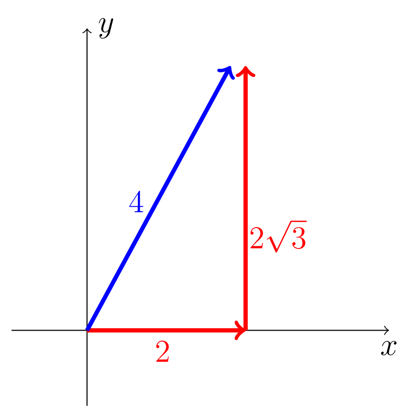
In physics, vectors are used to indicate a **displacement** of an object from one point A (the initial point) to another point B (the terminal point). For this we draw an arrow from A, the starting point, to B, the ending point. A vector is always symbolized by a lower-case letter with an arrow over it like $\vec{v}$. If it represents the displacement from A to B, then we can also write it as $\vec{AB}$.



We write this vector as

$$\vec{AB} = \vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

How is this the same as the math we saw earlier? This is just another way to define our vectors. We instead think of our vector as moving in the x direction then the y direction.



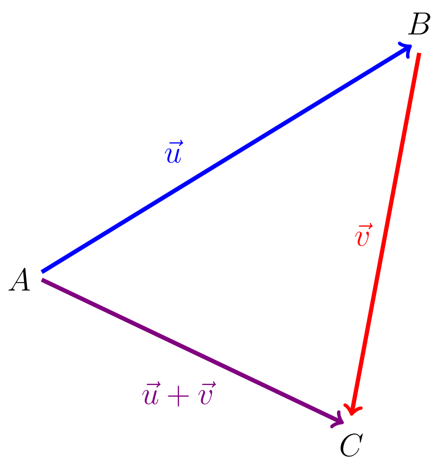
While this looks different, it is equivalent to our earlier formula. It's a matter of cartesian coordinates (traditional coordinates) versus polar coordinates.

Looking at this "cartesian" perspective allows us to see a way to describe the magnitude of a vector $\vec{v} = \langle x, y \rangle$, namely the **pythagorean theorem**. That is, the magnitude of $\vec{v}$, which is denoted as $|\vec{v}|$ is

$$|\vec{v}| = \sqrt{x^2 + y^2}$$

Magnitude can never be negative. The magnitude of a vector is only zero if the length of the arrow is 0; in that case, it looks more like a single point than an arrow, and there is no direction at all. This is called the zero vector, and it represents displacement from a point A to the same point A, or an object which is not moving. The notation for the zero vector is $\vec{0}$.

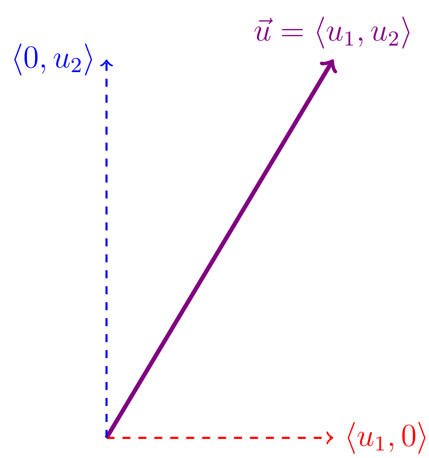
The sum of two vectors can be thought of as a combination of two displacements. For instance, if the vector $\vec{u}$ represents the displacement from point A to point B, and the vector $\vec{v}$ represents the displacement from the point B to point C, then their sum $\vec{u}+\vec{v}$ represents the total displacement from point A to point C. (This is called The Triangle Law.)



We add two vectors component-wise. For $\vec{u}=\langle x_1, y_1, \rangle$ and $\vec{v}=\langle x_2, y_2\rangle$,

$$\vec{u}+\vec{v}=\langle x_1+x_2, y_1+y_2\rangle$$

By this same logic, any vector can be broken up into its horizontal and vertical vectors. That is



Everything works the same way in three dimensions.

3.2.2 Examples

Example 3.2.2.1 *Construct the vector from point $A(1,3,5)$ to $B(-1,0,7)$*

To measure a total distance traveled, we want to subtract our final destination from our starting point.

$$\vec{AB} = \langle -1-1, 0-3, 7-5\rangle = \boxed{\langle -2,-3,2\rangle}$$

Example 3.2.2.2 *What is $\vec{v}+\vec{u}$ and $\vec{v}-\vec{u}$ for $\vec{v} = \langle 2,3,1\rangle$ and $\vec{u} = \langle -1,4,-5\rangle$*

Adding and subtracting vectors works component-wise. Be careful with your signs!

$$\begin{aligned}\vec{v}+\vec{u} &= \langle 2,3,1\rangle + \langle -1,4,-5\rangle \\ &= \langle 2+(-1), 3+4, 1+(-5)\rangle \\ &= \boxed{\langle 1,7,-4\rangle}\end{aligned}$$

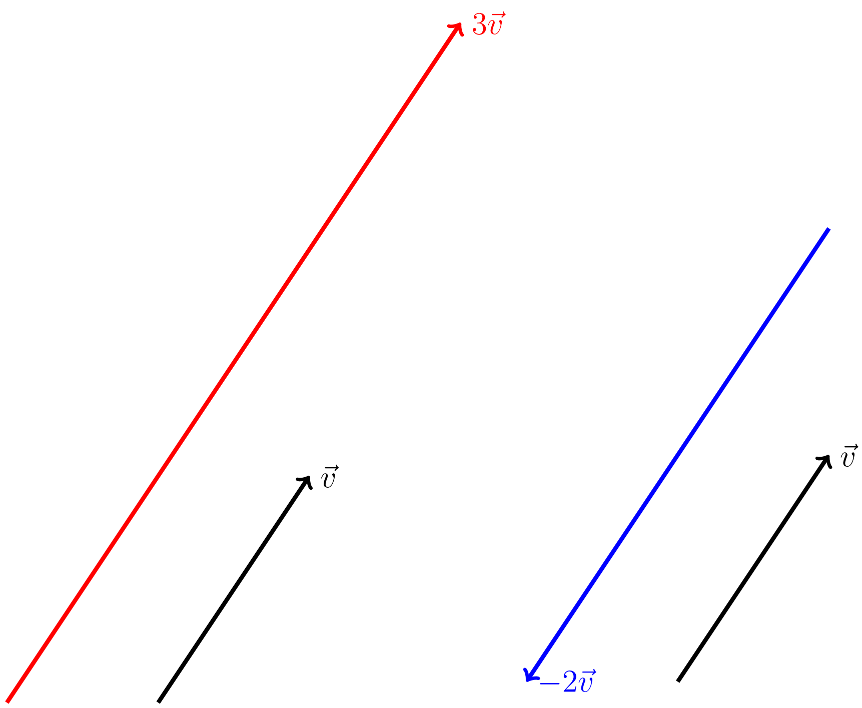
$$\begin{aligned}\vec{v}-\vec{u} &= \langle 2,3,1\rangle - \langle -1,4,-5\rangle \\ &= \langle 2-(-1), 3-4, 1-(-5)\rangle \\ &= \boxed{\langle 3,-1,6\rangle}\end{aligned}$$

Example 3.2.2.3 *What is the magnitude of $\vec{w} = \langle -2,1,-5\rangle$*

$$|\vec{w}| = \sqrt{(-2)^2+1^2+(-5)^2} = \sqrt{4+1+25} = \boxed{\sqrt{30}}$$

Constants, or just ordinary numbers, are called **scalars**. The magnitude of a vector is a scalar. There is also the notion of multiplying a vector by a scalar. For any vector $\vec{v}$ and any scalar c, the scalar multiple $c\vec{v}$ is obtained by multiplying the length of $\vec{v}$ by c and keeping the

- same direction if $c > 0$, or the
- opposite direction if $c < 0$.



Notice that multiplying by a scalar adjusts the magnitude, but doesn't rotate the arrow (it can only flip it).

In the same way we can focus on a **scalar component** of a vector (its magnitude), we can focus on just the direction of a vector. To do this, we consider a vector that points in the same direction but has length 1. This is called a **unit vector**.

3.2.3 Examples

Example 3.2.3.1 *What is $4\vec{v}$, where $\vec{v} = \langle 1, 1, 2\rangle$.*

To answer this, we just multiply each component by the scalar value.

$$\boxed{4\vec{v} = \langle 4, 4, 8\rangle}$$

Example 3.2.3.2 *Find the unit vector in the direction $\vec{v} = \langle 1, 2, 1\rangle$.*

How can we find a unit vector that points in the same direction?
First, let's find the length of $\vec{v}$.

$$|\vec{v}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

If we divide $\vec{v}$ by its magnitude, we get the vector

$$\vec{u} = \left(\frac{1}{\sqrt{6}}\right)\vec{v} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

Let's check our work. To do this, we ask ourselves, "What is the magnitude of this vector?"

$$|\vec{u}| = \sqrt{\left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{2}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2} = \sqrt{\frac{1}{6} + \frac{4}{6} + \frac{1}{6}} = 1$$

Notice that $\vec{u}$ is just the vector $\vec{v}$ multiplied by a positive scalar, so the direction doesn't change. So our unit vector is

$$\boxed{\vec{u} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle}$$

Summary of Ideas: Vectors

- A **vector** is an object that has **magnitude** (speed/length) and **direction**, but has no location attached to it. A vector describes *motion*!
- We use the pythagorean theorem to determine a vector's magnitude

  $$|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$$

  This is a scalar value.
- Using two points, we can construct a vector whose length is the distance between those points.
- We can add/subtract two vectors and multiply vectors by scalars.
- A **unit vector** is a vector of length one. From an arbitrary vector, $\vec{v}$, we can construct a unit vector in the same direction using the formula

  $$\vec{u} = \frac{\vec{v}}{|\vec{v}|}$$