4 Vector Functions

Now that we have a good grasp on three-dimensional mathematical objects, we will explore mathematical problems relevant to physics, like describing

- nonlinear paths,
- displacement,
- velocity,
- acceleration, and
- distance traveled.

Much of the math for this chapter will be based on the first course in calculus.

4.1 Vector Functions and Space Curves

Objectives

- ☐ I know what a vector function is, that is takes in scalar values, and produces vector values.
- ☐ I know how to find the domain of a vector function.

By now, we've seen *a lot* of different kinds of functions in three-dimensional space:

- lines
- cylinders
  - planes
- quadrics

For now, we will focus on our understanding of lines. Recall that we can describe lines with vector equations. Vector equations are those where the input is a scalar (like *time*) and the output is a vector. For example,

$$\vec{r}(t) = \langle 3 + 4t, -12 - t, 100 - 10t \rangle$$

represents a line.

This takes in a scalar, like $t = 0$, and spits out a vector,

$$\vec{r}(0) = \langle 3, -12, 100 \rangle$$

What if we put in functions more complicated than $3 + 4t$, like something nonlinear? Well, we'd no longer have a line, but we'd still have a **vector function**.

Definition 4.1.1 *A* **vector function** *is a function whose domain is made up of scalars (real numbers) and whose range is made up of vectors. Put another way, it takes in a scalar and produces a vector.*

We will focus on vector functions in three dimensions. This makes sense because that is how many dimensions we navigate in everyday life.

Our input will typically be t, which can be thought of as time. Again, this is sensible since our movements are recorded as values for particular times.

For general discussions, we will let $f(t)$ be the function giving the x-component, $g(t)$ be the function giving the y-component, and $h(t)$ be the function giving the z-component. So a vector function $\vec{r}$ can be written as

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

where f, g, and h are real-valued functions called the component functions of $\vec{r}(t)$.

Here is an example of a vector function:

$$\vec{r}(t) = \langle \ln(t), \cos(t), t \rangle$$

When we allow for more general functions, we no longer have a line. Instead we have some weird, curvy path in 3-dimensional space.

4.1.1 Examples

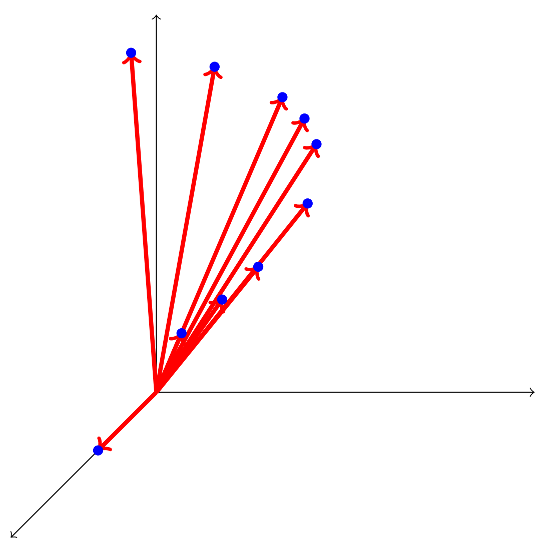
Example 4.1.1.1 *Graph the vector function*

$$\vec{r}(t) = \left\langle \cos(t), \sin(t), \sqrt{t} \right\rangle \qquad \textit{for} \;\; t > 0$$

For each value of t, $\vec{r}(t)$ produces a vector. We do not actually want to graph each vector because that would be hard to read. Instead, we imagine each vector beginning from the origin and pointing to a point. We graph that collection of points.

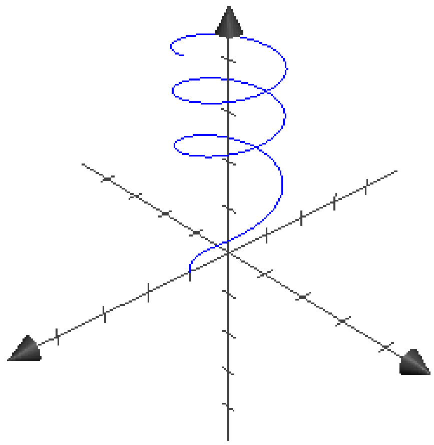
For example, if we considered the possible vectors of $\vec{r}(t)$, we get the following vectors:

t	$\vec{r}(t)$
0	$\langle 1, 0, 0 \rangle$
$\pi/6$	$\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{\frac{\pi}{6}} \right\rangle$
$\pi/4$	$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{\frac{\pi}{4}} \right\rangle$
$\pi/3$	$\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \sqrt{\frac{\pi}{3}} \right\rangle$
$\pi/2$	$\left\langle 0, 1, \frac{\sqrt{\pi}}{\sqrt{2}} \right\rangle$



Focus on the points in the picture above. The path we graph connects those points. That is, we graph and connect the *points* $(1, 0, 0)$, $(\sqrt{3}/2, 1/2, \sqrt{\pi/6})$, $(1/\sqrt{2}, 1/\sqrt{2}, \sqrt{\pi/4})$ and so on.

Our actual graph looks like this.



While we draw a path (pictured above), it is *very important we remember that each point represents a vector from the origin.*

The next natural goal is to find out the domains of these vector functions. The point of asking about the domain is to find out for what values of t will our description of motion be troublesome.

Given a vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ for some functions $f(t)$, $g(t)$, $h(t)$, the domain of r is every real number for which $f(t)$, $g(t)$, **AND** $h(t)$ are defined. The "and" is key. These have to be values of t that work for all three functions. That means you'll take the intersection of the three domains of each individual function.

4.1.2 Examples

Example 4.1.2.1 *What is the domain of the vector function*

$$\vec{r}(t) = \left\langle \ln(t), \frac{t^2+2}{(t-4)(t-2)}, \sqrt{t-1} \right\rangle$$

Let's consider the domain of each function.

- $\ln(t)$

  The domain is $t > 0$.

- $\dfrac{t^2+2}{(t-4)(t-2)}$

  The domain is $\{t \in \mathbb{R} : \ t \neq 4, t \neq 2\}$.

- $\sqrt{t-1}$

  The domain is $t \geq 1$.

So what points will work for *all three functions*?

If think think about it, $t \geq 1$ will work so long as $t \neq 2$ and $t \neq 4$. We can represent this domain in a few different ways.

- We can describe the intervals:
$$(1,2) \cup (2,4) \cup (4,\infty)$$
- We can also describe it with sets:

$$D = \{t \in \mathbb{R} : \ t \geq 1, t \neq 2, t \neq 4\}$$

Let's move on to limits, which will provide us with the tools to understand continuity and derivatives. We take the limit of a vector function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ as $t \to a$ by taking the limit of each of the component functions $f(t)$, $g(t)$, $h(t)$ as $t \to a$. In other words,

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle$$

The vector function $\vec{r}$ is continuous at $t = a$ if and only if each of the component functions f, g, and h are continuous at $t = a$. In other words, $\vec{r}(t)$ is continuous at $t = a$ when

$$\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$$

Put another way, when we can just plug in a value with no problem, the vector function is continuous at that value.

If our vector function is continuous, then the points $(f(t), g(t), h(t))$ for all values of t in the domain can be graphed in three-dimensional space to give us a curve, called a **space curve**.

4.1.3 Examples

Example 4.1.3.1 *Determine the limit*

$$\lim_{t\to\infty}\left\langle \frac{4t+3}{5t-1}, \frac{1}{t}, \frac{\sin(t)}{t} \right\rangle$$

To evaluate the limit, we simply evaluate the individual pieces.

$$\begin{aligned}
\lim_{t\to\infty}\left\langle \frac{4t+3}{5t-1}, \frac{1}{t}, \frac{\sin(t)}{t} \right\rangle &= \left\langle \lim_{t\to\infty}\frac{4t+3}{5t-1}, \lim_{t\to\infty}\frac{1}{t}, \lim_{t\to\infty}\frac{\sin(t)}{t} \right\rangle \\
&= \left\langle \lim_{t\to\infty}\frac{4}{5}, 0, 0 \right\rangle \\
&= \boxed{\left\langle \frac{4}{5}, 0, 0 \right\rangle}
\end{aligned}$$

Notice that for the x-component, we used L'Hôpital's rule. For the z component, we used sandwich theorem to see that

$$\lim_{t\to\infty}\frac{1}{t}\sin(t) = 0$$

Be sure to review the common techniques of determining limits.

Example 4.1.3.2 *Is the following function a space curve?*

$$\vec{r}(t) = \left\langle (t-3)^3, te^{t-2}, \cos(\pi t) \right\rangle$$

Yes, this is a space curve. To determine this, let's look at the domains.

- $(t-3)^3$

  The domain is all real numbers.

- te^{t-2}

  The domain is all real numbers.

- $\cos(\pi t)$

  The domain is all real numbers.

So all real numbers will work for all three functions. Therefore, the domain of $\vec{r}(t)$ can be represented as $(-\infty, \infty)$ or $\{t \in \mathbb{R}\}$ or simply $\mathbb{R}$.

Summary of Ideas: Vector Functions and Space Curves

- **Vector functions** are functions that take in a scalar value, like time, and produce a vector, like displacement.
- We graph them as if the vectors were points, but we remember that each point actually represents the motion beginning from the origin and moving towards that point.
- The domain of a vector function is the intersection of the domains of its individual components.
- Limits are also evaluated component-wise.
- A function is called a **space curve** when its domain is all real numbers.