

3.3 The Dot Product

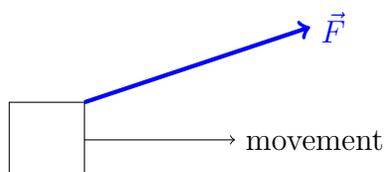
Objectives

- I can describe and mathematically define a dot product.
- I understand the connection between the dot product and orthogonality.
- I know what a scalar projection is and how to calculate it.
- I know what a vector projection is, how to draw it, and how to calculate it. I know its connection to a scalar projection.

Given two vectors, we might ask the question:

“To what extent do these two vectors go in the same direction?”

Consider a box being dragged horizontally by a force applied at an angle. Both its movement and its force are described using vectors. We are interested in the amount these two vectors share the same direction.



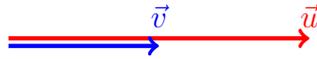
We will use this notion to understand the **dot product**. This will be a *scalar* value.¹ Before we go into the formulas, let’s talk a little bit about this idea of moving together.

If I have two perpendicular vectors, they don’t move in the same direction at all. These vectors are referred to as **orthogonal**. We will represent the amount they move together with 0. That is, $\vec{u} \cdot \vec{v} = 0$.



¹In fact, every notion of a *measurement* will be a scalar value.

Parallel vectors move *entirely* in the same direction. We will represent the amount they move together, which is the product of their magnitudes. That is, $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|$.



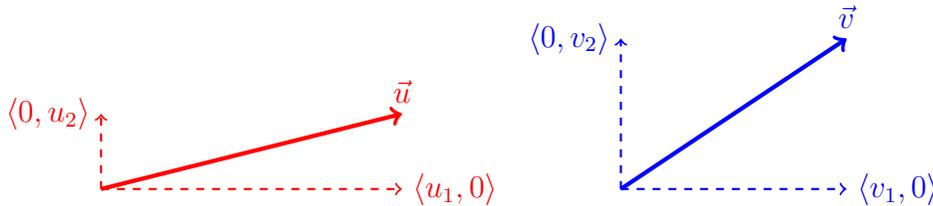
From these two examples, we can see that the angle between the two vectors plays a part. Since 0° corresponds to 1 and 90° corresponds to 0, we can deduce that our equation for the dot product is

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$$

where θ is the angle between them. Like in the last section, this is the polar coordinate version of a dot product and there is a cartesian version as well. For vectors $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$,

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2$$

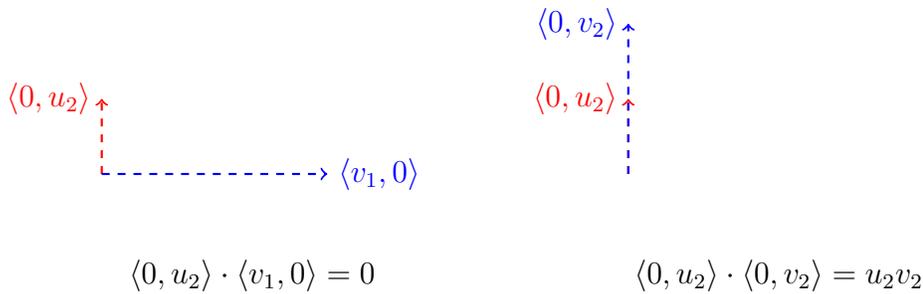
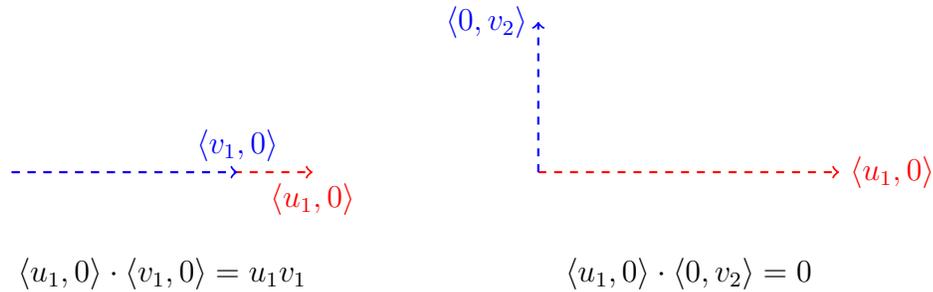
How might these two formulas be related? Let's think about the components of each vector.



Then we can FOIL out the separate components.

$$\vec{u} \cdot \vec{v} = \langle u_1, 0 \rangle \cdot \langle v_1, 0 \rangle + \langle u_1, 0 \rangle \cdot \langle 0, v_2 \rangle + \langle 0, u_2 \rangle \cdot \langle v_1, 0 \rangle + \langle 0, u_2 \rangle \cdot \langle 0, v_2 \rangle = u_1v_1 + u_2v_2$$

For the components that are parallel, we will get the product of the magnitudes. For the components that are perpendicular, we will get zero. Let's look at each pairing of vectors to see which is which.



Basically, when we break up a vector by its respective components and look at the dot product, we see why these two equations agree. The perpendicular components will have a zero dot product while the parallel components will only look at the product.

What happens when two parallel components point in opposite directions? For example, when

$$\langle -2, 0 \rangle \cdot \langle 3, 0 \rangle$$

Then the dot product can be expressed as either $| -2 ||3| \cos 180^\circ = -6$ and $(0) + (-2)(3) = -6$. Where did this 180° come from? If the vectors are parallel, then isn't the angle between them zero? No, it turns out the angle is measure when we place vectors from end to end. Remember, these objects have no location, so we can shift them anywhere.

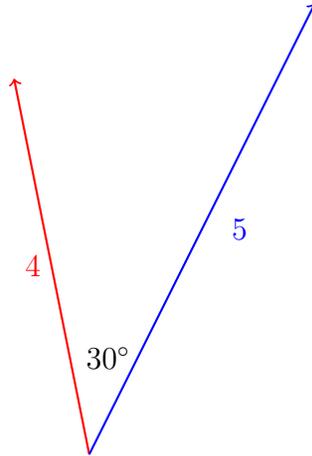


3.3.1 Examples

Example 3.3.1.1 1. What is the dot product of $\vec{u} = \langle 1, 2, -3 \rangle$ and $\vec{v} = \langle -1, 0, -1 \rangle$

$$\vec{u} \cdot \vec{v} = (1)(-1) + (2)(0) + (-3)(-1) = -1 + 3 = \boxed{2}$$

Example 3.3.1.2 2. *What is the dot product of two vectors pictured below.*



The dot product is

$$(4)(5) \cos 30^\circ = 10\sqrt{3}$$

With the use of dot products, we can talk about projections. There are two notions of a projection:

- a scalar projection
- a vector projection

The **scalar projection** of \vec{a} onto \vec{b} indicates the amount \vec{a} moves in the particular direction of \vec{b} . It is denoted $\text{comp}_{\vec{b}}\vec{a}$. The formula is

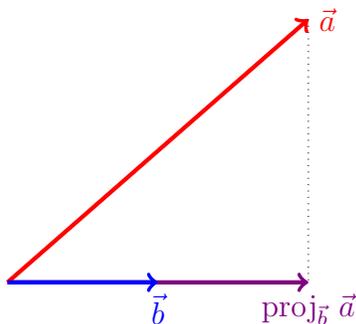
$$\text{comp}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

Why does this formula make sense? The dot product tells us the amount in which the two vectors move in the same direction. Because we only want a sense of how much \vec{a} moves, we divide by the magnitude of $|\vec{b}|$.

The **vector projection** of \vec{a} onto \vec{b} is a vector representative of the amount \vec{a} moves in the direction of \vec{b} . It is denoted $\text{proj}_{\vec{b}}\vec{a}$. The formula is $\text{comp}_{\vec{b}}\vec{a}$ times the **unit vector** in the direction of \vec{b} .

$$\text{proj}_{\vec{b}}\vec{a} = (\text{comp}_{\vec{b}}\vec{a}) \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$$

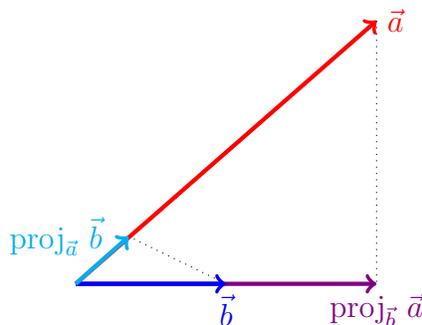
Notice that this is just the scalar projection multiplied by the unit vector in the direction of \vec{b} . This makes sense because the vector projection of \vec{a} onto \vec{b} should be a vector in the direction of \vec{b} whose magnitude reflects the amount at which \vec{a} points in the direction of \vec{b} .



One more thing to point out is that

$$\text{proj}_{\vec{b}} \vec{a} \neq \text{proj}_{\vec{a}} \vec{b}.$$

Let's look at a picture of the projection of $\text{proj}_{\vec{a}} \vec{b}$ to illustrate this fact.



The magnitude of our projection is $\text{comp}_{\vec{a}} \vec{b}$, so we can also deduce from this illustration that

$$\text{comp}_{\vec{b}} \vec{a} \neq \text{comp}_{\vec{a}} \vec{b}.$$

In other words, order matters!

3.3.2 Examples

Example 3.3.2.1 What is the angle between the vectors $\vec{a} = \langle 2, -2 \rangle$ and $\vec{b} = \langle 4, 0 \rangle$?

Remember that we have two notions of dot product. The first is that

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

and

$$\vec{a} \cdot \vec{b} = (2)(4) + (-2)(0) = 8$$

We can combine these to find

$$\frac{8}{|\vec{a}||\vec{b}|} = \cos \theta$$

Remember that $|\vec{a}| = \sqrt{(2)^2 + (-2)^2} = 2\sqrt{2}$ and $|\vec{b}| = \sqrt{(4)^2 + (0)^2} = 4$
Therefore,

$$\cos \theta = \frac{8}{(2\sqrt{2})(4)} = \frac{1}{\sqrt{2}} \implies \boxed{\theta = \frac{\pi}{4}}$$

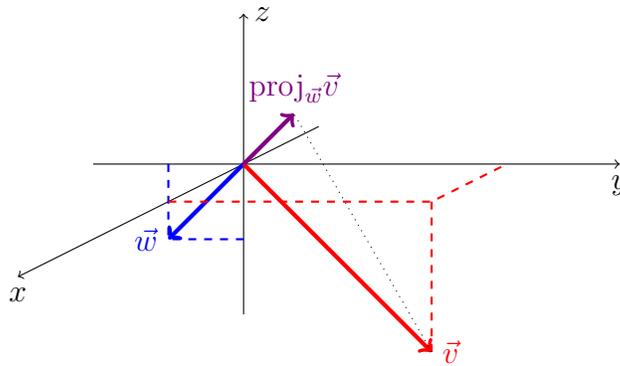
Example 3.3.2.2 Find the scalar and vector projections of \vec{v} onto \vec{w} where

$$\vec{v} = \langle 1, 3, -2 \rangle \quad \text{and} \quad \vec{w} = \langle 0, -1, -1 \rangle$$

The question asks for \vec{v} onto \vec{w} , so we're looking to find $\text{comp}_{\vec{w}}\vec{v}$ and $\text{proj}_{\vec{w}}\vec{v}$. When you sit down to memorize the formulas, remember that the vector you project onto will be the one that appears the most.

$$\text{comp}_{\vec{w}}\vec{v} = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} = \frac{(1)(0) + (3)(-1) + (-2)(-1)}{\sqrt{0^2 + (-1)^2 + (-1)^2}} = \boxed{-\frac{1}{\sqrt{2}}}$$

The scalar projection reflects the amount in which \vec{v} travels in the direction of \vec{w} . If the value is negative, it tells me that the vector $\text{proj}_{\vec{w}}\vec{v}$ is traveling in the opposite direction to \vec{w} .



Now, let's describe the vector pictured.

$$\text{proj}_{\vec{w}}\vec{v} = -\frac{1}{\sqrt{2}} \frac{\langle 0, -1, -1 \rangle}{\sqrt{2}} = -\frac{1}{2} \langle 0, -1, -1 \rangle = \boxed{\left\langle 0, \frac{1}{2}, \frac{1}{2} \right\rangle}$$

Summary of Ideas: Dot Product

- The **dot product** is a measurement of how much two vectors move together. We have two formulas for it.

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta \quad \text{and} \quad \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

- Combining these two formulas gives us a technique for determining the angle θ between two vectors.
- We can also use the dot product to measure the amount one vector (\vec{a}) moves in the direction of another vector (\vec{b}):

$$\text{comp}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

This is called the **scalar projection**.

- To find the **vector projection**, we take the scalar projection and multiply by the unit vector we are projecting onto.

$$\text{proj}_{\vec{b}}\vec{a} = \text{comp}_{\vec{b}}\vec{a} \frac{\vec{b}}{|\vec{b}|}$$