3.4   The Cross Product

Objectives

☐ I understand what a cross product produces and how to calculate it.

☐ I know how a cross product relates to the right-hand rule.

☐ I know how to find the area of a parallelogram and how to find the volume of a parallelepiped.

A cross product is the answer to three very important questions:

1. (**Algebra**) Can you "multiply" two vectors and get a third?

2. (**Geometry in 2D**) Two vectors can define a parallelogram. What is the area of this parallelogram?

3. (**Geometry in 3D**) Given two vectors in three-dimensional space, can we find a third vector perpendicular to them?

Let's address each of these questions individually to build our understanding of a cross product.

1. Algebra

So far, we know we can

- add two vectors to get a vector,

$$\langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

- multiply a scalar and a vector to get a vector,

$$c \langle x_1, y_1, z_1 \rangle = \langle cx_1, cy_1, cz_1 \rangle$$

- and dot two vectors to get a scalar.

$$\langle x_1, y_1, z_1 \rangle \cdot \langle x_2, y_2, z_2 \rangle = x_1 x_2 + y_1 y_2 + z_1 z_2$$

It seems natural to ask if we can "multiply" two vectors together to get a third. Surprisingly, in general the answer is no. Three dimensions just happen to be an exception.[2]

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[2]The only other exception is in seven dimensions, but it's a far less superior notion. So, one could argue that the only valuable exception is in three dimensions

2. Geometry in 2D

Two vectors define a parallelogram.
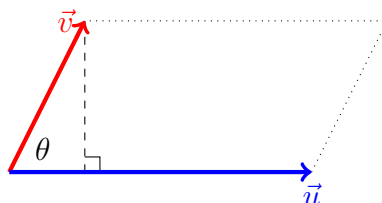
What is the area of that parallelogram? If you remember your geometry, you want the length of the base (in the picture above, that's $|\vec{u}|$) times the height (here, that's $|\vec{v}|\sin\theta$). That gives us
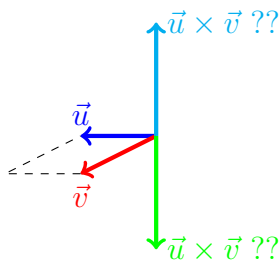
$$|\vec{u}||\vec{v}|\sin\theta$$

The problem is that this is a *scalar value*. So the cross product can't equal this; after all, we want a vector! What is the scalar component of a vector? It's magnitude! Therefore, our equation is
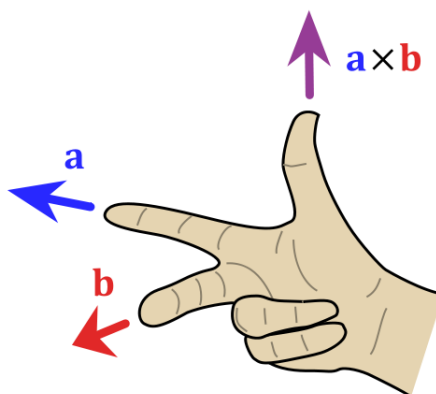
$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$$

3. Geometry in 3D

We want $\vec{u}\times\vec{v}$ to be perpendicular to both $\vec{u}$ and $\vec{v}$. Now, if I give you two vectors, you have two possible cross products.

Simply knowing the length and that is perpendicular is still a little ambiguous. Luckily, we already have a method to sort out the ambiguity in three dimensions: **the right-hand rule**.

Our index finger points along the first vector, our middle finger points along the second, and our thumb tells us the orientation. So our answer above is
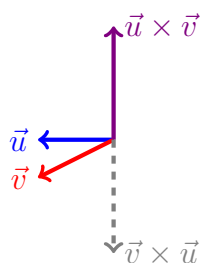
Now that we understand the conceptual notion of a cross product, let's look at some examples.

3.4.1 Examples

Example 3.4.1.1 *Let $\vec{a}$ and $\vec{b}$ be parallel vectors. What is $\vec{a} \times \vec{b}$?*
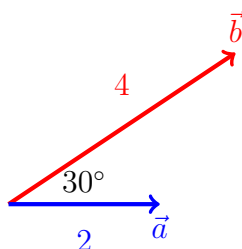
Let's begin by drawing a picture.

Notice that if $\vec{a}$ and $\vec{b}$ are parallel, then the area of the parallelogram they form is zero. Therefore, $\vec{a} \times \vec{b}$ must be zero as well.

$$\boxed{\vec{a} \times \vec{b} = 0}$$

Example 3.4.1.2 *Find $|\vec{a} \times \vec{b}|$ and determine if the cross product is coming in or out of the page.*

Using the right-hand rule, we know $\vec{a} \times \vec{b}$ comes $\boxed{\text{out of the page}}$. Furthermore,

$$|\vec{a} \times \vec{b}| = (2)(4) \sin \frac{\pi}{6} = \boxed{4}$$

Now that we have a clear picture of the concept, let's talk about the formula of the cross product. For vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$,

$$\vec{a} \times \vec{b} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = (a_2 b_3 - a_3 b_2)\vec{i} - (a_1 b_3 - a_3 b_1)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{i}$$

You can use the determinant definition above or you can memorize

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Why do we use a determinant to define this? Determinants are an expression of volume. By throwing $\vec{i}$, $\vec{j}$, and $\vec{k}$ at the top, we're imposing a vector structure. If instead we had a third vector, the determinant would reflect the volume of a **parallelpiped**.
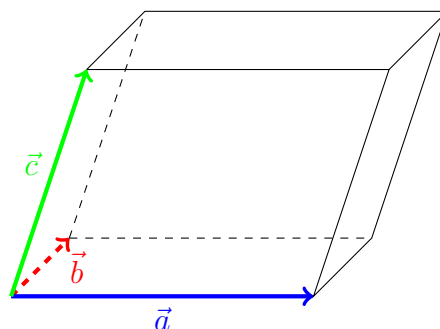
$$\vec{c} \cdot (\vec{a} \times \vec{b}) = \det \begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = c_1(a_2 b_3 - a_3 b_2) - c_2(a_1 b_3 - a_3 b_1) + c_3(a_1 b_2 - a_2 b_1)$$

You might have noticed that we can get a negative value with this formula, but volumes are positive. Therefore the volume is the absolute value of the determinant of this matrix. That is,

$$V = |\vec{c} \cdot (\vec{a} \times \vec{b})|$$

It's important to point out that this formula uses vectors, not points! The vectors represent the three different kinds of movements along the edges. So if you're given points to describe the figure, you'll need to construct the vectors that emanate from one point. (See example 3.4.2.2)

3.4.2   Examples

Example 3.4.2.1 *Let* $\vec{a} = \langle 6, 0, -1 \rangle$ *and* $\vec{b} = 1, 2, 0$. *What is* $\vec{a} \times \vec{b}$?

$$\vec{a} \times \vec{b} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix} = (0 - (-2))\vec{i} - (0 - (-1))\vec{j} + (12 - 0)\vec{i} = \boxed{\langle 2, -1, 12 \rangle}$$

Example 3.4.2.2 *The points* $P(1, 2, 1)$, $Q(1, 0, 0)$ *and* $R(0, 3, 1)$ *create a triangle. What is the area of the triangle?*

First, remember that points are *locations* while vectors are *movements*. We know the cross product relates to area and the cross product uses vectors. So, we have to (1) construct vectors and (2) use the cross product to define area.
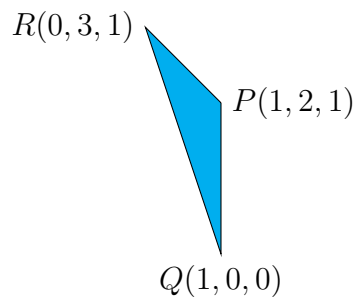
As we saw in the **Geometry in 2D** discussion, we need to use vectors oriented to point away from a single point that reflects a parallelogram. Let's define the vectors $\vec{PQ}$ and $\vec{PR}$. The cross product will be the area of the parallelogram, half of which is our triangle.

$$R(0, 3, 1) \qquad P(1, 2, 1)$$
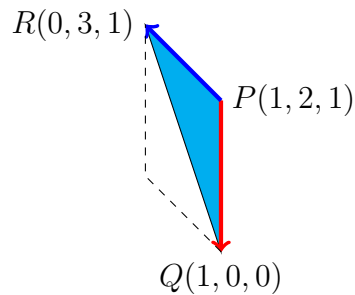
$$Q(1, 0, 0)$$

Hence, we calculate the cross product.

$$\vec{PQ} = \langle 1 - 1, 0 - 2, 0 - 1 \rangle = \langle 0, -2, -1 \rangle$$

$$\vec{PR} = \langle 0 - 1, 3 - 2, 1 - 1 \rangle = \langle -1, 1, 0 \rangle$$

$$\vec{PQ} \times \vec{PR} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & -1 \\ -1 & 1 & 0 \end{pmatrix} = (0 - (-1))\vec{i} - (0 - 1)\vec{j} + (0 - 2)\vec{i} = \langle 1, 1, -2 \rangle$$
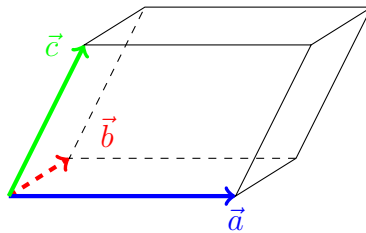
Now, to find the area of the parallelogram.

$$|\vec{PQ} \times \vec{PR}| = \sqrt{1 + 1 + 4} = \sqrt{6}$$

The area of the triangle is half the area of parallelogram. The final answer is $A = \boxed{\dfrac{\sqrt{6}}{2}}$

Example 3.4.2.3 *If $\vec{c} \cdot (\vec{a} \times \vec{b}) = 0$, must all three vectors be parallel? Explain.*

The expression $\vec{c} \cdot (\vec{a} \times \vec{b})$ reflects the volume spanned by the parallelepiped created by vectors $\vec{c}, \vec{a}, \vec{b}$. If a figure has zero volume, what must happen?

Remember that volume is length times width times height. If the volume is zero, then at least one of these values is zero.

The difference between length, width and height is a matter of viewpoint. So we can just assume height is what is zero. That means all three vectors lie within a flat plane.
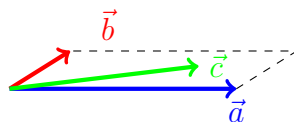
So our answer is **no**. These vectors aren't necessarily parallel. They are, however, always **coplanar**, meaning they all lie on the same plane.
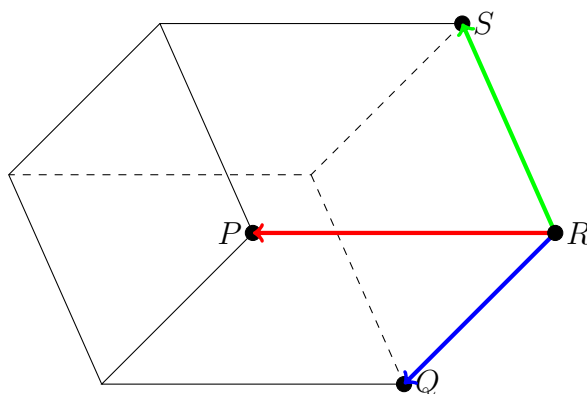
Example 3.4.2.4 *The points $P(1, 0, 0)$, $Q(1, 1, -1)$, $R(1, 2, 0)$ and $S(0, 1, 1)$ create a parallelepiped. What is its volume?*

First, to draw our picture, let's pick one point to be the pivot point. We will pick R, but we could have picked any point.
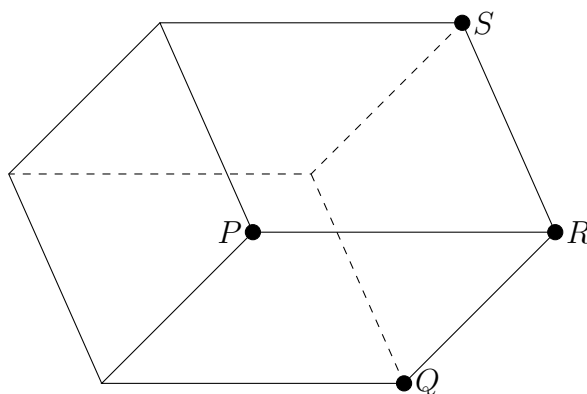
We have a formula but it uses vectors that describe the displacement between the points, not the actual points. So, we first need to define our vectors. They are

$$\vec{RS} = \langle 0 - 1, 1 - 2, 1 - 0 \rangle = \langle -1, -1, 1 \rangle$$
$$\vec{RP} = \langle 1 - 1, 0 - 2, 0 - 0 \rangle = \langle 0, -2, 0 \rangle$$
$$\vec{RQ} = \langle 1 - 1, 1 - 2, -1 - 0 \rangle = \langle 0, -1, -1 \rangle$$

Now, the volume will be the determinant of the matrix with these vectors as rows.

$$\vec{RS} \cdot (\vec{RP} \times \vec{RQ}) = \det \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \end{pmatrix} = -1(2 - 0) + 1(0 - 0) + 1(0 - 0) = -2$$

The final answer is $\boxed{V = 2}$

Summary of Ideas: Cross Product

- Cross products "multiply" two vectors and give you a third.

- The magnitude of a cross product reflects the area of a parallelogram created your input vectors. The formula is

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin \theta$$

- The cross product is perpendicular to the two vectors given and follows the right-hand rule.

- The formula for the cross product is

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

- The scalar $|\vec{c} \cdot (\vec{a} \times \vec{b})|$ equals the volume of the parallelepiped created by all three vectors.