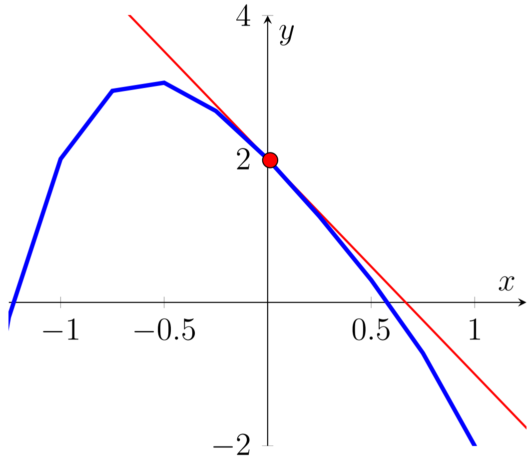
5.5 Tangent Planes and Linear Approximations

Objectives

- ☐ I understand the difference between the function $f(x, y) = z$ and the function $F(x, y, z) = f(x, y) - z$.
- ☐ I can calculate ∇f and ∇F.
- ☐ I can use ∇F to define a tangent plane.
- ☐ Once I have a tangent plane, I can calculate the linear approximation.

Tangent lines are used to approximate complicated surfaces. For example,



While we may not know the blue function pictured above, if we know the point $(0, 2)$ and that $f'(0) = -3$, then we know the function of the tangent line (red). That is, we know the tangent line is the function

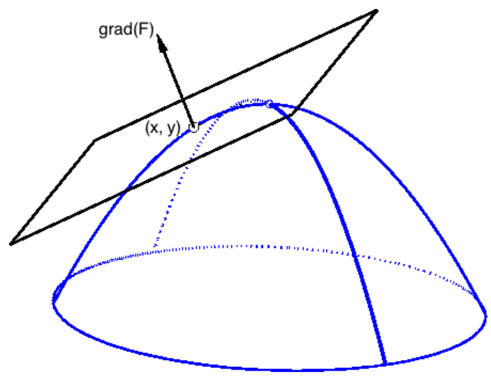
$$y = -3x + 2$$

If you were asked to estimate $f(0.5)$, you could use the line to do so. You'd guess

$$f(0.5) \approx -3(0.5) + 2 = 0.5$$

This method is called **linear approximation** because we make an estimate using a line. In the example above, the blue line is actually $f(x) = -x^4 + x^3 - x^2 - 3x + 2$. In that case, $f(0.5) = 0.5625$, which is pretty close to our estimate of 0.5. This turns out to be a very useful mathematical tool, especially in the sciences.

Just as two-dimensional curves have a tangent line at each point, three-dimensional surfaces have **tangent planes** at each point. We can use this tangent plane to make approximations of values close by the known value.



From our work in the section "Lines and Planes," we know a plane is defined by a normal vector and a point. The natural question that follows is,"How do I find this normal vector on a surface?" We use the **gradient**.

Recall that the **gradient vector** is the vector whose entries are partial derivatives. For example,

$$(\nabla f)(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

We sometimes use $grad(f)$ to denote ∇f. Both notations are common.

Note, however, that ∇f cannot be our normal vector. It points in the direction of steepest ascent and has no height (it only has two dimensions). Therefore, it can't be normal to the plane. Why? Imagine you're out hiking. You and your friends are on a path and you ask your friend to point toward the steepest hill around you. Your friend is not pointing to the sky! She's point in a direction parallel to the xy-plane.

To find the normal vector, we will consider the gradient of a related function. Specifically,

$$F(x, y, z) = f(x, y) - z$$

Its gradient,

$$grad(F) = \nabla F = \langle F_x, F_y, F_z\rangle = \langle f_x, f_y, -1\rangle$$

is always **perpendicular to the function** $f(x, y)$**!**

Why is it perpendicular? Just like ∇f points in the direction of steepest ascent in the xy-plane, ∇F points toward steepest ascent in the xyz-graph. When F increases, the surface dilates outward (think of blowing up a balloon; as you add air, the surface grows outward). Because ∇F points towards the growth, it is a perpendicular vector that points outward. To find the tangent plane of

$f(x, y)$ at the point (a, b), we will

1. find $c = f(a, c)$,
2. find $F(x, y, z)$,
3. calculate $\nabla F(a, b, c)$, and
4. plug into our equation for the plane

$$n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$$

5.5.1 Example

Example 5.5.1.1 *Find the tangent plane at point P.*

$$f(x, y) = xy^2, \;\; P(2, 1)$$

Approximate the value of $f(2.1, 1.5)$.

Let's follow the steps.

1. $c = f(2, 1) = 2$
2. $F(x, yz) = xy^2 - z$
3. $(\nabla F)(x, y, z) = \langle y^2, 2xy, -1\rangle$ so

$$(\nabla F)(2, 1, 2) = \langle 1, 4, -1\rangle$$

4. plug into our equation for the plane

$$(x - 2) + 4(y - 1) - (z - 2) = 0 \implies x + 4y - z = 4$$

Since we will use this plane for approximations on z, we should write $\boxed{z = x + 4y - 4}$
Now that we have our equation, we can approximate $f(2.1, 1.5)$ using our plane. That is

$$f(2.1, 1.5) \approx 2.1 + 6 - 4 = \boxed{4.1}$$

Example 5.5.1.2 *Find the tangent plane at point P.*

$$f(x,y) = \frac{y^2}{x}, \; P(1,2)$$

Approximate the value of $f(2,2)$.

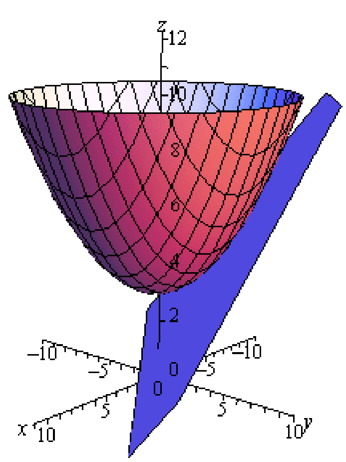
Let's follow the steps.

1. $c = f(1,2) = 4$
2. $F(x,yz) = \frac{y^2}{x} - z$
3. $$(\nabla F)(x,y,z) = \left\langle -\frac{y^2}{x^2}, \frac{2y}{x}, -1 \right\rangle$$
   so $(\nabla F)(1,2,4) = \langle -4, 4, -1 \rangle$
4. plug into our equation for the plane
   $$-4(x-1) + 4(y-2) - (z-4) = 0 \implies -4x + 4y - z = 0$$

Again, we should write $\boxed{z = -4x + 4y}$
Now that we have our equation, we can approximate $f(2,2)$ using our plane. That is

$$f(2,2) \approx -8 + 8 = \boxed{0}$$

An important thing to note is that approximations are only meaningful *near the point where the plane is tangent*. The further away from the original point you go, the worse your approximation.



Sometimes, you will be asked to approximate z with only the values of the partial derivatives. We'll use the same formula as before, but let's put it in a more accessible form.

First, remember that our normal vector at a point (a, b) will always end up as

$$\langle f_x(a,b), f_y(a,b), -1\rangle$$

We can plug this into the formula for the plan and solve for z.

$$z \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

With this identity, we can approximate z when we are given values.

Let's look at two examples.

5.5.2 Examples

Example 5.5.2.1 *Given that f is a differentiable function with $f(2,5) = 6$, $f_x(2,5) = 1$, and $f_y(2,5) = -1$, use a linear approximation to estimate $f(2.2, 4.9)$.*

We'll just plug in, recognizing that $(a, b) = (2, 5)$. That gives us the equation

$$z \approx 6 + (x-2) - (y-5)$$

Now that we have our equation, we just plug in.

$$6 + (2.2-2) - (4.9-5) = 6 + .2 - (-.1) = 6.3$$

Hence, our answer is $\boxed{f(2.2, 4.9) \approx 6.3.}$

Example 5.5.2.2 *Given that f is a differentiable function with $f(1,1) = 1$, $f_x(1,1) = 3$, and $f_y(1,1) = 4$, use a linear approximation to estimate $f(1.2, 0.9)$.*

Let's begin my finding the gradient at (1, 1).

Our equation is

$$z \approx 1 + 3(x-1) + 4(y-1)$$

We plug in to get

$$z \approx 1 + 3(0.2) + 4(0.1) = 1 + 0.6 + 0.4 = 2$$

Hence, $\boxed{f(1.2, 0.9) \approx 2.}$

Finally, you may be asked to estimate the *change* in z given changes in x and y. We can use the formula

$$z \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

by subtracting $f(a,b)$ from both sides. Since $z - f(a,b) = \Delta z$, we get

$$\Delta z \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$$

Let's see some examples.

5.5.3 Examples

Example 5.5.3.1 *Suppose at a point p that $f_x = 2$ and $f_y = -1$. If x is perturbed by .5 units in the positive direction and y is perturbed by .3 units in the negative direction, how much is z perturbed and in what direction?*

Using the formula, we get

$$\Delta z \approx 2\Delta x - \Delta y = 2(.5) - (-.3) = 1.3$$

Therefore, $\boxed{\Delta z \approx 1.3.}$

Example 5.5.3.2 *For the function*

$$f(x,y) = xy^3,$$

calculate the gradient at the point $(2,1)$ *and estimate* Δz *if* $\Delta x = 1$ *and* $\Delta y = 2$.

Let's begin my finding the partial derivatives at (2, 1).

- $f_x(x,y) = y^3$, which implies $f_x(2,1) = 1$
- $f_y(x,y) = 3xy^2$, which implies $f_y(2,1) = 6$

Now, we may plug into the equation.

$$z \approx \Delta x + 6\Delta y = 1 + 6(2) = 13$$

Hence, $\boxed{\Delta z \approx 13.}$

Summary of Ideas: Tangent Planes and Linear Approximations

- The gradient of a function is normal to its level curves.
- For a function $z = f(x, y)$, we can define the function $F(x, y, z) = f(x, y) - z$. Then, ∇F is *perpendicular* to the surface of $f(x, y)$.
- We can use ∇F to find the **tangent plane** at any point (a, b) on the surface. First, we find $c = f(a, b)$. Then we get the plane

$$0 = F_x(x - a) + F_y(y - b) + F_z(z - c)$$

- This plane can be used for **linear approximations**. To do this, we write our plane in $z =$ form and plug in to approximate values. The closer your point is to (a, b) (the point where the plane is tangent), the better your approximation.
- We also use two other equations, which are equivalent to the tangent plane:

$$z \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

  This formula is useful to think of when you are just given values.
- The other formula tests changes in variables. It is written

$$\Delta z \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y$$