MATH 231: Calculus of Several Variables
Section 1, 107 Ag Sc & Ind Bldg,
TR 9:05 AM - 9:55 AM

---

1 Reparametrization With Respect to Arc Length

We begin with our familiar formula for arc length. Given a vector function $\vec{r}\,'(t)$, we can calculate the length from $t = a$ to $t = b$ as

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

We can actually turn this formula into a function of time. That is, we can create a function $s(t)$ that measures how far we've traveled from $\vec{r}(a)$ at time t. We replace b with t. To keep things clear, let's use the extra variable τ.

$$s(t) = \int_a^t |\vec{r}\,'(\tau)|\, d\tau$$

We can use this function to reparametrize $\vec{r}(t)$ so that our input is the distance we traveled from $t = a$.

To clarify, let's look at some examples.

Example 1.0.0.1 *Reparametrize*

$$\vec{r}(t) = \langle 2t, 3\sin(2t), 3\cos(2t)\rangle$$

by its arc length starting from $(0, 0, 3)$. *Use this information to determine the position after traveling* $\pi\sqrt{10}$ *units.*

First, let's figure out for what value of t will we get the point $(0, 0, 3)$. We can do this by setting each entry equal to the corresponding value. First, we consider,

$$2t = 0 \implies t = 0$$

If $t = 0$, then we need $3\sin(0) = 0$ and $3cos(0) = 3$, which are both true. So, this happens at $t = 0$. This is our initial value.

Now, let's figure out $s(t)$.

First,

$$\vec{r}\,'(t) = \langle 2, 6\cos(2t), -6\sin(2t)\rangle$$

We now plug in

$$\begin{aligned} s(t) &= \int_0^t \sqrt{4 + 36\cos^2(2\tau) + 36\sin^2(2\tau)}d\tau \\ &= \int_0^t \sqrt{4+36}d\tau \\ &= \int_0^t \sqrt{40}d\tau \\ &= t\sqrt{40} \end{aligned}$$

Once we have our expression

$$s = t\sqrt{40}$$

we can solve for t. When we do, we get

$$t = \frac{s}{\sqrt{40}}$$

We then plug into $\vec{r}(t)$ to get our final answer.

$$\vec{r}(s) = \left\langle \frac{2s}{\sqrt{40}}, 3\sin\left(\frac{2s}{\sqrt{40}}\right), 3\cos\left(\frac{2s}{\sqrt{40}}\right)\right\rangle = \boxed{\left\langle \frac{s}{\sqrt{10}}, 3\sin\left(\frac{s}{\sqrt{10}}\right), 3\cos\left(\frac{s}{\sqrt{10}}\right)\right\rangle}$$

Finally, we plug in the distance $\pi\sqrt{10}$ to determine the location.

$$\vec{r}(\pi\sqrt{10}) = \left\langle \frac{\pi\sqrt{10}}{\sqrt{10}}, 3\sin\left(\frac{\pi\sqrt{10}}{\sqrt{10}}\right), 3\cos\left(\frac{\pi\sqrt{10}}{\sqrt{10}}\right)\right\rangle = \langle \pi, 0, -3\rangle$$

Example 1.0.0.2 *Reparametrize*

$$\vec{r}(t) = \langle -\sin(t), \cos(t), 1\rangle$$

by its arc length starting from $(-1, 0, 1)$.

We're going to do the same steps as the previous problem. We begin with figuring out for what value of t will we get the point $(-1, 0, 1)$. We can do this by setting each entry equal to the corresponding value. First, we consider,

$$-\sin(t) = -1 \implies t = \frac{\pi}{2}$$

If $t = \pi/2$, then we need $\cos(\pi/2) = 0$, which is true. So, this is our initial value.

Now, let's figure out $s(t)$.
Let's calculate the derivative.

$$\vec{r}\,'(t) = \langle -\cos(t), -\sin(t), 0\rangle$$

We now plug in

$$\begin{aligned} s(t) &= \int_{\pi/2}^{t} \sqrt{\cos^2(\tau) + \sin^2(\tau)}d\tau \\ &= \int_{\pi/2}^{t} \sqrt{1}d\tau \\ &= t - \pi/2 \end{aligned}$$

Once we have our expression

$$s = t - \pi/2$$

we can solve for t. When we do, we get

$$t = s + \pi/2$$

We then plug into $\vec{r}(t)$ to get our final answer.

$$\vec{r}(s) = \langle -\sin(s + \pi/2), \cos(s + \pi/2), 1\rangle = \boxed{\langle -\cos(s), \sin(s), 1\rangle}$$

1.1 Practice Problems

1. Reparametrize
$$\vec{r}(t) = \langle 2t + 1, 4t, 3t - 1\rangle$$
by its arc length starting from $(3, 4, 2)$. Use this information to determine the position after traveling 5 units.

2. Reparametrize
$$\vec{r}(t) = \langle -\sin(3t), -\cos(3t), 4t\rangle$$
by its arc length starting from $(0, -1, 0)$. Use this information to determine the position after traveling π units.

3. Reparametrize
$$\vec{r}(t) = \left\langle \sin\left(\frac{t}{2}\right), \cos\left(\frac{t}{2}\right), \frac{t}{\sqrt{2}}\right\rangle$$
by its arc length starting from $(0, 1, 0)$.

4. Reparametrize

$$\vec{r}(t) = \langle r\sin(t), r\cos(t), 0\rangle$$

by its arc length starting from $(0, r, 0)$.