5.3   Partial Derivatives

> **Objectives**
>
> ☐ I know how to take a partial derivative with respect to a variable.
>
> ☐ I understand the notation for partial derivatives.
>
> ☐ I can use Clairaut's Theorem to make my calculations easier.

The notion of limits and continuity are relevant in defining **derivatives**. When a function has more than one variable, however, the notion of derivative becomes vague. We no longer simply talk about a derivative; instead, we talk about a **derivative with respect to** a variable. The remaining variables are fixed. We call this a **partial derivative**.

To denote the specific derivative, we use subscripts. For example, the derivative of f with respect to x is denoted f_x.

Let's look at some examples.

For the following examples, the color blue will indicate a portion of the function that is treated as a constant. Think of these portions as being *frozen*. The portions that have changed (because of a derivative) are in red.

14.3.1 Examples

Example 5.3.0.4 1. *Find the first partial derivatives of the function*

$$f(x,t) = e^{-t}\cos(\pi x)$$

Since there is only two variables, there are two first partial derivatives. First, let's consider f_x. In this case, t is fixed and we treat it as a constant. So, e^{-t} is just a constant.

$$\boxed{f_x(x,t) = -e^{-t}\pi\sin(\pi x)}$$

Now, find f_t. Here, x is fixed so $\cos(\pi x)$ is just a constant.

$$\boxed{f_t(x,t) = -e^{-t}\cos(\pi x)}$$

Example 5.3.0.5 2. *Find the first partial derivatives of the function*

$$f(x, y) = x^4 y^3 + 8x^2 y$$

Again, there are only two variables, so there are only two partial derivatives. They are

$$\boxed{f_x(x, y) = 4x^3 y^3 + 16xy}$$

and

$$\boxed{f_y(x, y) = 3x^4 y^2 + 8x^2}$$

Higher order derivatives are calculated as you would expect. We still use subscripts to describe the second derivative, like f_{xx} and f_{yy}. Interestingly, we can get mixed derivatives like f_{xy} and f_{yx}. In addition, we know

$$f_{xy} = f_{yx}$$

regardless of our choice of f. This is called **Clairaut's Theorem**. What's the point of knowing this theorem? It means that you can switch the order of derivatives based on whatever would be easiest.

Clairaut's Theorem extends to higher derivatives. If we were looking at taking two derivatives with respect to x and one with respect to y, we would have three possible ways to do this

$$f_{yxx} = f_{xyx} = f_{xxy}$$

You may have heard of **partial differential equations**. These are equations that use derivatives of an unknown function as variables. The goal is to try to figure out the original function. For example, our understanding of waves is based on partial differential equations. Specifically, we look at something called **the wave equation**

$$u_{tt} = a^2 u_{xx}.$$

Let's look at some example problems on partial derivatives and partial differential equations.

14.3.2 Examples

Example 5.3.0.6 1. *Find f_{xxx}, f_{xyx} for*

$$f(x,t) = \sin(2x + 5y)$$

Let's begin by finding f_x and use that to find f_{xx} and f_{xxx}

$$f_x = 2\cos(2x + 5y)$$

Remember that $5y$ is just treated as a constant. Notice that we could work towards finding f_{xyx} by finding f_{xy} from the above equation. If we use Clairaut's Theorem, however, we can skip a step by calculating f_{xxy} instead. Now, let's calculate f_{xx}.

$$f_{xx} = 2\left(-2\sin(2x + 5y)\right) = -4\sin(2x + 5y)$$

Using f_{xx}, we can find f_{xxx} and f_{xxy}. They are

$$\boxed{f_{xxx} = -8\cos(2x + 5y)}$$

and

$$\boxed{f_{xyx} = f_{xxy} = -20\cos(2x + 5y)}$$

Example 5.3.0.7 2. *Find f_{xyz} for*
$$f(x, y, z) = e^{xyz^2}$$

This is a good example to pay close attention to because it illustrates how complicated these partial derivatives can get.

Let's first find f_x. It is

$$f_x = yz^2 e^{xyz^2}$$

Notice the coefficients. Because y and z are treated as constants, they need to be brought out front by the chain rule. For the next derivative, we will have to use the product rule. What does this tell us? It tells us that it's probably better to take f_z first since we won't get that pesky z^2.

$$f_z = 2zxye^{xyz^2}$$

Notice that taking the derivative with respect to x or y next will result in the same amount of work. Let's just pick x next.

$$f_{zx} = (2zxy)(yz^2 e^{xyz^2}) + (2zy)(e^{xyz^2}) = 2xy^2 z^3 e^{xyz^2} + 2zy e^{xyz^2}$$

The parentheses are in place to indicate how I broke up the variables to take the derivatives. Now let's calculate the last derivative, the partial derivative with respect to y.

$$f_{zxy} = (2z)(e^{xyz^2}) + (2zy)(xz^2 e^{xyz^2}) + (2xy^2 z^3)(xz^2 e^{xyz^2}) + (4xyz^3)(e^{xyz^2})$$

After we simplify, we get the final answer

$$\boxed{f_{zxy} = 2z e^{xyz^2} \left[1 + 3xyz^2 + x^2 y^2 z^4 \right]}$$

Example 5.3.0.8 3. *Show that $u = \sin(kx)\sin(akt)$ is a solution to the wave equation*

$$u_{tt} = a^2 u_{xx}.$$

To do this, we need to find u_{tt} and u_{xx} and show that the equation holds.

$$u_t = ak \sin(kx) \cos(akt)$$
$$\implies u_{tt} = -a^2 k^2 \sin(kx) \sin(akt)$$

$$u_x = k \cos(kx) \sin(akt)$$
$$\implies u_{tt} = -k^2 \sin(kx) \sin(akt)$$

Plugging into the wave equation, we get

$$[u_{tt}] = a^2 [u_{xx}]$$
$$\implies \left[-a^2 k^2 \sin(kx) \sin(akt) \right] = a^2 \left[-k^2 \sin(kx) \sin(akt) \right]$$
$$\implies -a^2 k^2 \sin(kx) \sin(akt) = -a^2 k^2 \sin(kx) \sin(akt)$$

Since our resulting equation is trivially true, then we know $u = \sin(kx)\sin(akt)$ is a solution to the wave equation.

Summary of Ideas: Partial Derivatives

- A partial derivative with respect to a variable, takes the derivative of the function with respect to *that variable* and treats *all other variables as constants.*

- The order in which we take partial derivatives does not matter. That is, $f_{xyz} = f_{yzx} = f_{zyx} = f_{yxz} = f_{zxy} = f_{xzy}$.

- We can determine if a function is a solution to a partial differential equation by plugging it into the equation.