

4.4 Motion in Space: Velocity and Acceleration

Objectives

- I know what displacement, velocity, and acceleration are.
- I know how to calculate speed.
- I know what the osculating plane is.
- I can define acceleration as a sum of two vectors: the normal vector and the tangent vector. I can find the coefficients for each vector in that sum in two different ways.

For this section, we will focus on how these mathematical tools are used in physics.

We begin with a vector function $\vec{r}(t)$, which will describe the displacement of an object at time t . The velocity of this object is

$$\vec{v}(t) = \vec{r}'(t)$$

An important component to the object's velocity is its speed, which corresponds to the magnitude of the vector function. That is, the speed is $|\vec{v}(t)|$.

Similarly, the object's acceleration is

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$$

4.4.1 Examples

Example 4.4.1.1 Find the velocity, acceleration, and speed of a particle with the given position function.

$$\vec{r}(t) = \langle t^2, 2t, \ln(t) \rangle$$

Then find these for $t = 2$

Velocity is

$$\vec{v}(t) = \vec{r}'(t) = \left\langle 2t, 2, \frac{1}{t} \right\rangle$$

Acceleration is

$$\vec{a}(t) = \vec{v}'(t) = \left\langle 2, 0, -\frac{1}{t^2} \right\rangle$$

Speed is

$$v(t) = |\vec{v}(t)| = \sqrt{(2t)^2 + (2)^2 + \frac{1}{t^2}} = \frac{\sqrt{4t^4 + 4t^2 + 1}}{t} = \frac{\sqrt{(2t^2 + 1)^2}}{t} = 2t + \frac{1}{t}$$

At $t = 2$,

$$\vec{v}(2) = \left\langle 4, 2, \frac{1}{2} \right\rangle$$

$$\vec{a}(2) = \left\langle 2, 0, -\frac{1}{4} \right\rangle$$

$$v(2) = 4 + \frac{1}{2} = 4.5$$

We can relate $\vec{a}(t)$ to the unit tangent vector $\vec{T}(t)$ and the unit normal vector $\vec{N}(t)$ of the vector function $\vec{r}(t)$. To show this, we need to define the speed function, which takes in time and produces speed (a scalar value).

$$v(t) = |\vec{v}(t)| = |\vec{r}'(t)|$$

We can use this function to relate $\vec{v}(t)$ to $\vec{T}(t)$.

$$\vec{r}'(t) = \vec{v}(t) = v(t)\vec{T}(t)$$

This holds because $\vec{T}(t)$ always points in the direction of $\vec{r}'(t)$ and $v(t)$ is always its magnitude. We can take the derivative and use the product rule.

$$\vec{a}(t) = \vec{v}'(t) = \frac{d}{dt} [v(t)\vec{T}(t)] = v'(t)\vec{T}(t) + v(t)\vec{T}'(t)$$

Let us now recall two facts.

1.

$$\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$$

2.

$$\kappa = \frac{|\vec{T}'|}{|\vec{r}'|} = \frac{|\vec{T}'|}{v} \implies \kappa v = |\vec{T}'|$$

Therefore,

$$\vec{T}'(t) = |\vec{T}'| \vec{N} = \kappa v \vec{N}$$

We combine everything to get

$$\vec{a}(t) = v'(t)\vec{T}(t) + \kappa v^2(t)\vec{N}(t)$$

We call $a_T = v'(t)$ and $a_N = \kappa v^2(t)$.

Lets study this formula. First, observe that we don't use the binormal vector \vec{B} . That means acceleration lies entirely in the plane spanned by \vec{T} and \vec{N} (called **the osculating plane**). Recall that \vec{T} gives the direction the object is moving and \vec{N} gives the direction the curve is turning.

Next we notice that the tangential component of acceleration is v' , the rate of change of speed, and the normal component of acceleration is κv^2 , the curvature times the square of the speed. This makes sense if we think of a passenger in a car a sharp turn in a road means a large value of the curvature κ , so the component of the acceleration perpendicular to the motion is large and the passenger is thrown against a car door. High speed around the turn has the same effect; in fact, if you double your speed, a_N is increased by a factor of 4.

There are alternate formulas for these coefficients that are much easier to calculate; however, I won't derive them. Here they are

$$a_T = \frac{\vec{v}(t) \cdot \vec{a}(t)}{|\vec{v}(t)|} \quad a_N = \frac{|\vec{v}(t) \times \vec{a}(t)|}{|\vec{v}(t)|}$$

4.4.2 Examples

Example 4.4.2.1 Find the tangential and normal components of the acceleration vector.

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

Velocity is

$$\vec{v}(t) = \langle -\sin t, \cos t, 1 \rangle$$

Acceleration is

$$\vec{a}(t) = \langle -\cos t, -\sin t, 0 \rangle$$

The tangent vector is

$$\vec{T}(t) = \frac{1}{\sqrt{\cos^2 t + \sin^2 t + 1}} \langle -\sin t, \cos t, 1 \rangle = \left\langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

The coefficient is

$$a_T = \frac{\vec{v}(t) \cdot \vec{a}(t)}{|\vec{v}(t)|} = \frac{\sin t \cos t - \sin t \cos t + 0}{\sqrt{2}} = 0$$

So the tangential component is $\vec{0}$.

The normal vector is

$$\vec{N}(t) = \frac{\sqrt{2}}{1} \left\langle -\frac{\cos t}{\sqrt{2}}, -\frac{\sin t}{\sqrt{2}}, 0 \right\rangle = \langle -\cos t, -\sin t, 0 \rangle$$

The coefficient is

$$a_N = \frac{|\vec{v}(t) \times \vec{a}(t)|}{|v(t)|} = \frac{1}{\sqrt{2}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \left| \left\langle \frac{\sin t}{\sqrt{2}}, -\frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \right| = 1$$

So the normal vector is $\langle -\cos t, -\sin t, 0 \rangle$.

Notice that in this example, the acceleration is always perpendicular to the path. This is always true for circular paths. Think about when you are in a car traveling in a circle. You lean toward the center because that is where the acceleration pulls you.

Summary of Ideas: Motion in Space: Velocity and Acceleration

- If $\vec{r}(t)$ is a displacement vector, then velocity is

$$\vec{v}(t) = \vec{r}'(t)$$

and acceleration is

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$$

- Acceleration is a sum of the tangent vector $\vec{T}(t)$ and the normal vector $\vec{N}(t)$. That is, $\vec{a} = a_T \vec{T}(t) + a_N \vec{N}(t)$. The coefficients are

$$a_T = v' \quad \text{and} \quad a_N = \kappa v^2$$

- Alternatively, we can define the coefficients as

$$a_T = \frac{\vec{v}(t) \cdot \vec{a}(t)}{|\vec{v}(t)|} \quad a_N = \frac{|\vec{v}(t) \times \vec{a}(t)|}{|\vec{v}(t)|}$$