5.7 Maximum and Minimum Values

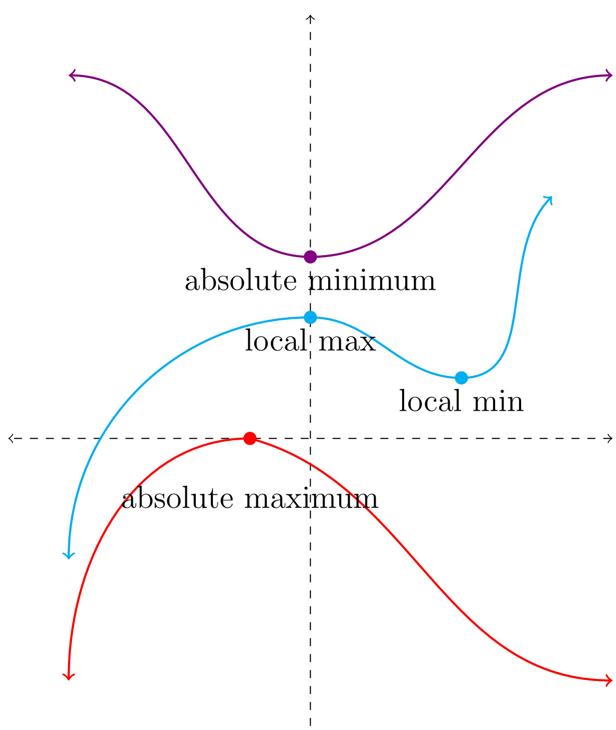
Objectives

- ☐ I can define critical points.
- ☐ I know the difference between local and absolute minimums/maximums.
- ☐ I can find local maximum(s), minimum(s), and saddle points for a given function.
- ☐ I can find absolute maximum(s) and minimum(s) for a function over a closed set D.

In many physical problems, we're interested in finding the values (x, y) that maximize or minimize $f(x, y)$.

Recall from your first course in calculus that **critical points** are values, x, at which the function's derivative is zero, $f'(x) = 0$. These x-values either maximized $f(x)$ (a **maximum**) or minimized $f(x)$ (a **minimum**).

We did not simply call a critical point a maximum or a minimum, however. Sometimes your critical point is local, meaning it's not the highest/lowest value achieved by the function, but it's the highest/lowest point "near by." See the image below for clarification.

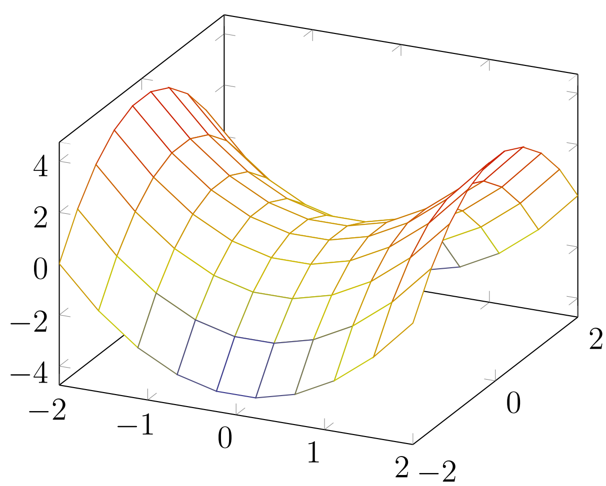


To classify critical points as maximums or minimums, we look at the second derivative. The point was called a minimum if $f''(x_0) > 0$ and it was called a maximum if $f''(x_0) < 0$. I like the mnemonic, "concave up (+) is like a cup; concave down (-) is like a frown."

For functions of two variables, $z = f(x, y)$, we do something similar.

Definition 5.7.1 *A point (a, b) is a* **critical point** *of $z = f(x, y)$ if the gradient, ∇f, is the zero vector.*

Critical points in three dimensions can be maximums, minimums, or saddle points. A **saddle point** mixes a minimum in one direction with a maximum in another direction, so it's neither (see the image below).



Once a point is identified as a critical point, we want to be able to classify it as one of the three possibilities. Like you did in calculus, you will look at the second derivative. In higher dimensions, this is the determinant of a matrix containing *all possible* second derivatives. This matrix is called the **Hessian matrix**.

$$\det \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

If you are not comfortable with matrices, you may may memorize the following formula

$$d = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

The critical point is classified by the value of D.

- If $d > 0$ and $f_{xx}(a,b) > 0$, then the point is a **(local) minimum**. For f_{xx} we can think, "concave up (+) is like a cup."
- If $d > 0$ and $f_{xx}(a,b) < 0$, then the point is a **(local) maximum**. For f_{xx} we can think, "concave down (-) is like a frown."
- If $d < 0$, then the point is a **saddle point**.
- If $d = 0$, then the test fails and the point could be anything.

5.7.1 Examples

Example 5.7.1.1 *Find and classify all critical values for the following function.*

$$f(x,y) = xy - 2x - 2y - x^2 - y^2$$

First, we need to find the zeros of the partial derivatives. Those partials are

- $f_x(x,y) = y - 2 - 2x$
- $f_y(x,y) = x - 2 - 2y$

Set both of these partial derivatives to zero.

- $0 = y - 2 - 2x$
- $0 = x - 2 - 2y$

Then we solve the system of equations.

$$x = 2 + 2y \implies y = 2 + 2(2 + 2y) \implies y = 2 + 4 + 4y$$

Then $-3y = 6$ gives us that $y = -2$. We can plug in to find x

$$x = 2 + 2(-2) = -2$$

The solution is $\boxed{(-2,-2).}$ That is our critical point.

Now, we need to classify it. Let's find the second partial derivatives:

- $f_{xx}(x, y) = -2$
- $f_{yy}(x, y) = -2$
- $f_{xy}(x, y) = 1$

Then

$$d = (-2)(-2) - 1 = 3$$

Since $d = 3 > 0$ and $f_{xx} = -2 < 0$, then we have a $\boxed{\text{local maximum.}}$

Example 5.7.1.2 *Find and classify all critical values for the following function.*

$$f(x, y) = x^3 - 12xy - 8y^3$$

First, we need to find the zeros of the partial derivatives. Those partials are

- $f_x(x, y) = 3x^2 - 12y$
- $f_y(x, y) = -12x - 24y^2$

Set both of these partial derivatives to zero.

- $y = (1/4)x^2$
- $x = -2y^2$

Next, we solve the system of equations.

$$y = (1/4)(-2y^2)^2 \implies y = y^4$$

If $y = 1$, then $x = -2$. If $y = 0$, then $x = 0$. The critical points $\boxed{(-2, 1)}$ and $\boxed{(0, 0).}$

We now calculate the second derivatives to classify the critical point.

- $f_{xx}(x, y) = 6x$
- $f_{yy}(x, y) = -48y$
- $f_{xy}(x, y) = -12$

Then

$$d = (6x)(-48y) - (12)^2 = -288xy - 144$$

Now, let's determine d for the point $(-2, 1)$.

$$d(-2, 1) = 576 - 144 = 432 > 0$$

Since $d(-2, 1) = 432 > 0$ and $f_{xx}(-2, 1) = 6(-2) = -12 < 0$, then $(-2, 1)$ a $\boxed{\text{local maximum.}}$

$$d(0, 0) = 0 - 144 < 0$$

Thus, $(0, 0)$ is a $\boxed{\text{saddle point.}}$

Example 5.7.1.3 *Find and classify all critical values for the following function.*

$$f(x, y) = y\cos(x)$$

As before, we begin by finding the partial derivatives:

- $f_x(x, y) = -y\sin(x)$
- $f_y(x, y) = \cos(x)$

Set both of these equations equal to zero.

- $0 = -y\sin(x)$
- $0 = \cos(x)$

For these equations to hold, we have the following conditions:

- $y = 0$, or $x = \cdots - \pi, 0, \pi, \cdots$
- $x = \cdots - 3\pi/2, -\pi/2, \pi/2, 3\pi/2, \cdots$

The only way to make both partial derivatives zero is to choose

- $x = \cdots - 3\pi/2, -\pi/2, \pi/2, 3\pi/2, \cdots$
- $y = 0$

Therefore, the critical values are

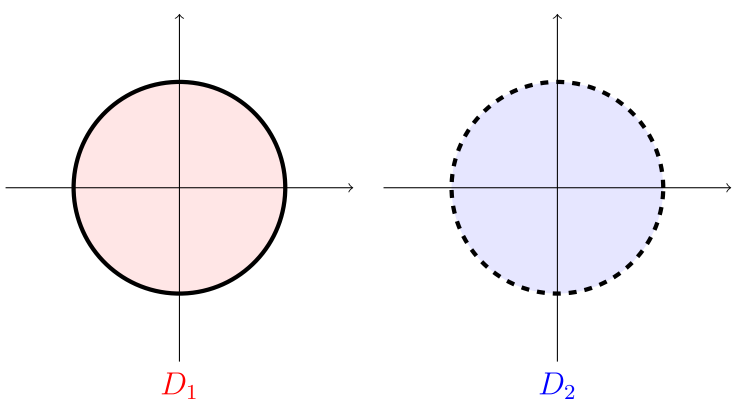
$$\cdots(-3\pi/2, 0), (-\pi/2, 0), (\pi/2, 0), (3\pi/2, 0)\cdots$$

Now, we want to classify these points. To do that, we will calculate d.

- $f_{xx}(x, y) = -y\cos(x)$
- $f_{xy}(x, y) = -\sin(x)$
- $f_{yy}(x, y) = 0$

Then
$$d = 0 - \sin^2(x) \leq 0$$
For the points,
$$\cdots(-3\pi/2, 0), (-\pi/2, 0), (\pi/2, 0), (3\pi/2, 0)\cdots$$
d is never zero. Therefore, $d < 0$ and that means all these critical points are $\boxed{\text{saddles.}}$

If the domain is a closed set, then f has an **absolute minimum** and an **absolute maximum**. A **closed set** is a set that includes its boundary points. For example $D_1 = \{(x, y)|x^2 + y^2 \leq 2\}$ is closed because it includes its boundary while $D_2 = \{(x, y)|x^2 + y^2 < 2\}$ is not closed because it does not.



To find the absolute maximum and absolute minimum, follow these steps:

1. Find the the critical points of f on D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

The most challenging part of these problems will be considering the values of f on the boundary. You will reduce the problem in one of two ways. Let's consider the domain D_1 above and try to maximize $f(x, y) = xy$ on its boundary.

- One method is to solve one variable in terms of another. The boundary is $2 = x^2 + y^2$, so we could solve and say $y = \pm\sqrt{2 - x^2}$. Then we can plug in for y to get $f(x, y) = f(x) = \pm x\sqrt{2 - x^2}$. The boundary's critical points are precisely those values of x for which

  $$0 = f'(x) = \mp\frac{2(x^2 - 1)}{\sqrt{2 - x^2}}$$

  This is only true when $x = \pm 1$. We then find the corresponding values of y and find the extreme points on the boundary are

  $$(1, 1), (1, -1), (-1, -1), \text{ and } (-1, 1)$$

- Alternatively, we could parameterize the boundary. That means we pick $x = \sqrt{2}\sin t$ and $y = \sqrt{2}\cos t$. Then we get $f(t) = x(t)y(t) = 2\sin t\cos t = \sin 2t$. We find the critical points of $f(t)$ by solving $0 = f'(t) = 2\cos 2t$. Here, we need to consider all values of t between 0 and 2π because that is a full rotation around the boundary. Therefore, this is true if $t = \pi/4, 3\pi/4, 5\pi/4$, and $7\pi/4$. That means our extreme points are $(\sqrt{2}\sin t, \sqrt{2}\cos t)$ for those values of t. That is,

  $$(1, 1), (1, -1), (-1, -1), \text{ and } (-1, 1)$$

Either approach will work. I recommend you use the one that makes the most sense to you in the given problem.

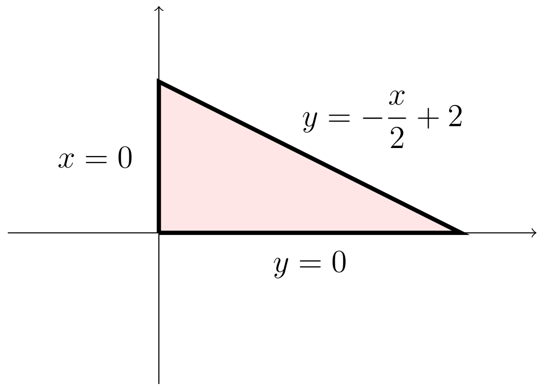
Before we look at examples, let's briefly discuss terminology. When we say "critical points," we mean points where the derivative or gradient equals zero ($f'(x) = 0$ or $\nabla f = \vec{0}$). We use the term extreme value to just mean the biggest or smallest. The distinction is that an extreme value may not make the derivative zero, but it still may give the largest value.

5.7.2 Examples

Example 5.7.2.1 *Find the absolute maximum and minimum values of the function on D, where D is the enclosed triangular region with vertices* $(0, 0), (0, 2)$, *and* $(4, 0)$.

$$f(x, y) = x + y - xy$$

Let's first draw a picture of D to help us visualize everything.



First, we find the critical points on D. We begin by finding the partials and setting them equal to zero

- $f_x(x, y) = 1 - y = 0$
- $f_y(x, y) = 1 - x = 0$

The only critical point on D is $(1, 1)$. Notice that $f(1, 1) = 1$

Now, we find the extreme points on the boundary. We will use the information in our picture to help us.

From $(0, 0)$ to $(0, 2)$, the line is $x = 0$. We will parameterize this line as $(0, 2t)$ for $0 \leq t \leq 1$. We can plug in these values to get

$$f(0, 2t) = 0 + 2t - (0)(2t) = 2t$$

The maximum value is 2, which is achieved at $(0, 2)$. The minimum value is 0, which is achieved at $(0, 0)$.

From $(0, 0)$ to $(4, 0)$, the line is $y = 0$. We parametrize this as $(4t, 0)$ for $0 \leq t \leq 1$. Along this line, the values are

$$f(4t, 0) = 4t$$

The maximum value is 4, which is achieved at $(4, 0)$. The minimum value is 0, which is achieved at $(0, 0)$.

From $(4, 0)$ to $(0, 3)$, the line that defines it is $y = -x/2 + 2$ for $0 \leq x \leq 4$. Instead or parametrizing, let's plug in.

$$f(x) = x + \left(-\frac{x}{2} + 2\right) - x\left(-\frac{x}{2} + 2\right) = \frac{x^2}{2} - \frac{3x}{2} + 2$$

The critical points are the values of x such that

$$0 = f'(x) = x - \frac{3}{2}$$

The critical point is then $(3/2, 5/4)$ and $f(3/2, 5/4) = -1$. We do not need to check the end points since we already know those values.

Now, let's take a moment to study all the critical points we've found:

- $f(1, 1) = 1$
- $f(0, 2) = 2$
- $f(0, 0) = 0$
- $f(4, 0) = 4$
- $f(3/2, 5/4) = -1$

Therefore, the absolute maximum happens at $\boxed{(4, 0)}$ and the absolute minimum happens at $\boxed{(3/2, 5/4).}$

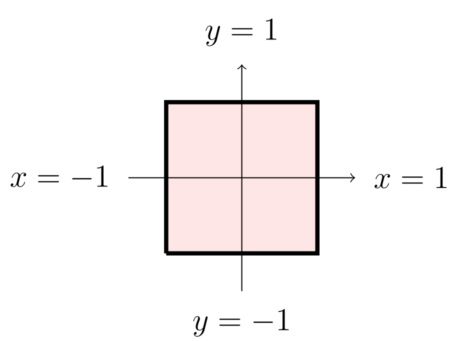
Example 5.7.2.2 *Find the absolute maximum and minimum values of f on the set D, where*

$$f(x, y) = x^2 + y^2 + x^2y + 4$$

and

$$D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$$

We start with the same process as before. While it's not required, it's always good to start with a picture of your domain.



Let us first find the critical points in D. We solve the following equations

- $f_x(x, y) = 2x + 2xy = 2x(1 + y) = 0$
- $f_x(x, y) = 2y + x^2 = 0$

The critical points are therefore $(0,0)$ and $(\sqrt{2},-1)$. We should take a moment to observe that $f(0,0)=4$ and $f(\sqrt{2},-1)=5$.

Now, let's consider the extreme points of the boundary.

Let's begin with the section $x=-1$. Then,

$$f(y)=1+y^2+y+4=y^2+y+5$$

This function achieves its critical point when

$$f'(y)=2y+1=0 \implies y=-1/2$$

so at $(-1,-1/2)$ is a critical point on the boundary. Let's take note that $f(-1,-1/2)=19/4$ or we can think of this as 4.75. It's also important to check the end points, since these are extreme points on the boundary. When we do, we get $f(-1,-1)=5$ and $f(-1,1)=7$.

Now let's check $x=1$. Then,

$$f(y)=1+y^2+y+4=y^2+y+5$$

By the exact same work above, we know $(1,-1/2)$ will be a critical point for the boundary. We, again, take note that $f(1,-1/2)=19/4$. We also need to check the corners because these are extreme points: $f(1,-1)=5$ and $f(1,1)=7$.

At this point, we've looked at all the corners. For the two remaining boundary lines, we can skip this step.

Let us now check $y=1$. This gives us

$$f(x)=x^2+1+x^2+4=2x^2+5$$

If we consider its derivative, $f'(x)=4x$, we see that we have a critical point at $x=0$. We should check that $f(0,1)=4$.

Finally, we check $y=-1$. This gives us

$$f(x)=x^2+1-x^2+4=5$$

This is a flat line, so it has no critical points.

Now, let's tally all the points we found.

Critical points of the Surface in D

- $f(0,0)=4$
- $f(\sqrt{2},-1)=5$

Critical points on the boundary

- (On $x=-1$): $f(-1,-1/2)=19/4=4.75$

- (On $x = 1$): $f(1, -1/2) = 19/4$
- (On $y = 1$): $f(0, 1) = 4$

Corner Values

- $f(-1, -1) = 5$
- $f(-1, 1) = 7$
- $f(1, -1) = 5$
- $f(1, 1) = 7$

Therefore, our absolute maximum is at $\boxed{(-1, 1) \text{ and } (1, 1)}$ and our absolute minimum is at $\boxed{(-1, -1/2) \text{ and } (1, -1/2).}$

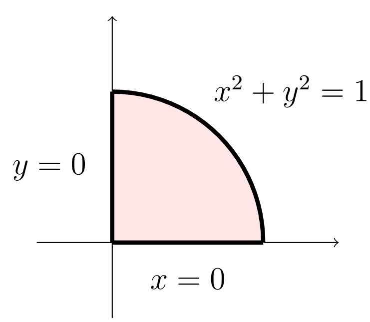
Example 5.7.2.3 *Find the absolute maximum and minimum values of f on the set D, where*

$$f(x, y) = xy^3$$

and

$$D = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$$

This domain will be just a quarter of a circle since we restrict both x and y to be greater than zero.



Let us first find the critical points of the function.

- $f_x(x, y) = y^3 = 0$
- $f_y(x, y) = 3xy^2 = 0$

Then $y = 0$ is a sufficient condition to get a critical point. That is, all points $(x, 0)$ are critical! To stay in our domain, we'll consider $0 \le x \le \sqrt{1-y^2} = 1$.

$$f(x,0) = 0$$

Now, let's consider the boundary. You may have noticed that we already considered $y = 0$ above. Let's look at $x = 0$.

$$f(0,y) = 0$$

This doesn't have critical points, and all values are equal to zero (like on $y = 0$). While this may be a frustrating result, we *know* this problem must have a maximum and minimum. Let's check the last portion of the boundary.

To do this portion, we will parametrize. Let $x = \cos t$, $y = \sin t$, and let $0 \le t \le \pi/2$. Then

$$f(t) = \cos t \sin^3 t$$

Let's find the critical points for the function.

$$\begin{aligned} f(t) &= 3\cos^2 t \sin^2 t - \sin^4 t \\ &= \sin^2 t(3\cos^2 t - \sin^2 t) \\ &= \sin^2 t(2\cos^2 t + \cos^2 t - \sin^2 t) \\ &= \sin^2 t(2\cos^2 t + 2\cos^2 t - 1) \\ &= \sin^2 t(4\cos^2 t - 1) \end{aligned}$$

Remember that $\sin^2 t = 0$ when $t = 0$, which gives us the point $(1, 0)$. This point also lives on the piece $y = 0$, so we've already looked at it.

We are primarily interested in $(4\cos^2 t - 1) = 0$. That's why we did so much work with trigonometric manipulations. For what value(s) of t will this happen?

$$4\cos^2 t - 1 = 0 \implies \cos^2 t = 1/4 \implies t = \pi/3$$

At $t = \pi/3$, we have the point $(\cos \pi/3, \sin \pi/3) = (1/\sqrt{2}, 3/\sqrt{2})$. Then

$$f(1/\sqrt{2}, 3/\sqrt{2}) = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{3}{\sqrt{2}}\right)^3 = \frac{27}{4}$$

Therefore the absolute minimum values occur at all points of the form $\boxed{(x,0) \text{ and } (0,y)}$. The absolute maximum occurs at $\boxed{(1/\sqrt{2}, 3/\sqrt{2}).}$

Summary of Ideas: Maximum and Minimum Values

- Critical points are those points (x, y) such that $\nabla f(x, y) = 0$.
- They are classified as local maximums, minimums and saddles using the determinant of the **Hessian matrix**.

$$d = \det \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

- We classify according the following rules:
  - If $d > 0$ and $f_{xx}(a,b) > 0$, then the point is a **(local) minimum**.
  - If $d > 0$ and $f_{xx}(a,b) < 0$, then the point is a **(local) maximum**.
  - If $d < 0$, then the point is a **saddle point**.
  - If $d = 0$, then the test fails and the point could be anything.
- Over a closed region D, you can find the absolute minimum and absolute maximum. These are the smallest and largest values achieved by $f(x, y)$, respectively.
- To find these values, we first find the critical points on D. We then restrict f to the boundary of D and find the extreme values. We can solve one variable in terms of another and plug in the expression or we can parametrize the path and plug in $(x(t), y(t))$.