MATH 231: Calculus of Several Variables
Section 1, 107 Ag Sc & Ind Bldg,
TR 9:05 AM - 9:55 AM

1 The Philosophy of this Class

Before we begin, let's spend some time discussing the philosophy behind this course, what you can expect, and what is expected of you.

1.1 Philosophy: Mindsets and Learning Math

Many people believe that a person is either good at math or not. Put in another way, you need a certain kind of brain to be good at math. This is an idea that one sees on television all the time; math is talked about as if it is hard, frustrating and completely unrewarding. The underlying message is that it's OK to not do well in math because it's impossibly difficult, just for 'geeks,' and some people just can't do it.

This viewpoint is called a "fixed mindset." It's a perspective that's really only prevalent in a few countries including the US and the UK. In most other countries, this is an unheard of notion. It doesn't even make sense. Scientific research has shown that human brains have the ability to change physically, functionally, and chemically *throughout your life*. This is called **brain plasticity**. There have been stunning examples of the brain's ability to create new neurons and connections in recent years. In one case, a girl completely compensated for having the right half of her brain removed within weeks and gained full motor function of her entire body.

Every time we learn, new neurons and connections between neurons form in response to that new information. Think of it as trails in a forest. The paths that are well traveled are cleared and easy to see year after year; without use, however, the forest grows back and they disappear. What we do, what we think, and how we feel all influence our brain.

An interesting study that illustrated this looked at the brains of London cab drivers. They have to learn over 300 routes, 25,000 streets, and 20,000 landmarks and points of interest for their profession. They have to pass a test called "The Knowledge." Once you've passed this, you can work anywhere in the greater London area; it takes between 2 to 4 years for most to pass this exam. Researchers studied the brains of these cab drivers and found that their hippocampus, the part of the brain responsible for memory and spacial navigation, grew to be significantly bigger and more developed than the average person. Once they retired, this part of their brain shrank back down again.

What does this tell us? All students are capable of achieving at the highest levels in math (and really anything else). This more realistic perspective is called the "growth mindset."

Interestingly, many of the world's top mathematicians echo this perspective.

There is no Nobel Prize in mathematics, but the equivalent is probably the Fields Medal. It's an award given out every 4 years to up to four people. I'd like to share with you the following quote by one awardee:

> I was always deeply uncertain about my own intellectual capacity. I thought I was unintelligent. And it's true that I was, and I still am, rather slow. I need time to seize things because I always need to understand them fully. Even when I was the first to answer the teacher's questions, I knew it was because they happened to be questions to which I already knew the answer. But if a new question arose, usually students who weren't as good as I was answered before me. Towards the end of the 11th grade I secretly thought of myself as stupid .... I never talked about this to anyone, but I always felt convinced that my imposture would someday be revealed; the whole world and myself would see that what looked like intelligence was really just an illusion. If this ever happened, apparently no one noticed it, and I'm still just as slow... I had real trouble taking notes; it's still difficult for me to follow a seminar.
>
> At the end of the eleventh grade I took measure of the situation and came to the conclusion that **rapidity doesn't have a precise relationship to intelligence. What is important is to deeply understand things and their relations to each other. This is where intelligence lies.** The fact of being quick or slow isn't really relevant. Naturally, it's helpful to be quick, like it is to have a good memory. But it's neither necessary nor sufficient for intellectual success.
>
> –Laurent Schwarz, Fields Medal winner

From this information (and so much more), you should know that:

1. **You can master this material.** Everyone here is capable of excelling in this class; your potential far exceeds the material we will cover here.

2. **Ability is not a fixed quality.** Struggling in previous classes is not indicative of anything; your brain is capable of growing all the tools you need so long as you have the right materials at your disposal.

3. **Learn the concepts deeply.** Time is no longer an issue if you've already learned a concept deeply. If we really understand these concepts, the exams will be a breeze.

1.1.1 What's Your Mindset?

Those who have a **fixed mindset** often

- hate to fail and
- avoid challenging work at all costs.

Alternatively, those with a **growth mindset** are

- persistent,
- learn from mistakes, and
- are encouraged by the success of others.

Fixed mindsets affect students at all levels regardless of how well they typically do in a class. It's important to take a moment and recognize which mindset best describes you. Having a more growth-oriented view will mean more success in your academic career and beyond. Employers universally prefer employees who are willing to work at a problem until they can solve it; only a growth mindset is equipped to do that.

Learning from your mistakes might be the best skill you can have. *Research has shown that those who make mistakes and correct them learn material better than those who never made mistakes at all.*

1.2 Course Structure: What You Can Expect

Your grade for this course is broken down into three parts:

- Homework
- Effort
- Midterm Exam
- Final Exam

The goal of the homework assignments and classroom activities (the effort portion) are meant to be **formative** assessments, or assessments for learning. Their primary goal is to instruct.

Your midterm and final, of which I have no control, are designed to be **summative** assessments, or assessments meant to measure your knowledge of the subject. Their primary goal is measurement.

1.2.1 Homework

You will be assigned homework after every class. Because we learn best from mistakes, I will give everyone an opportunity to re-do any or all problems marked incorrect by a specified time. I *highly* encourage that you do this.

1. **Assignments.** After every class, you are assigned homework.

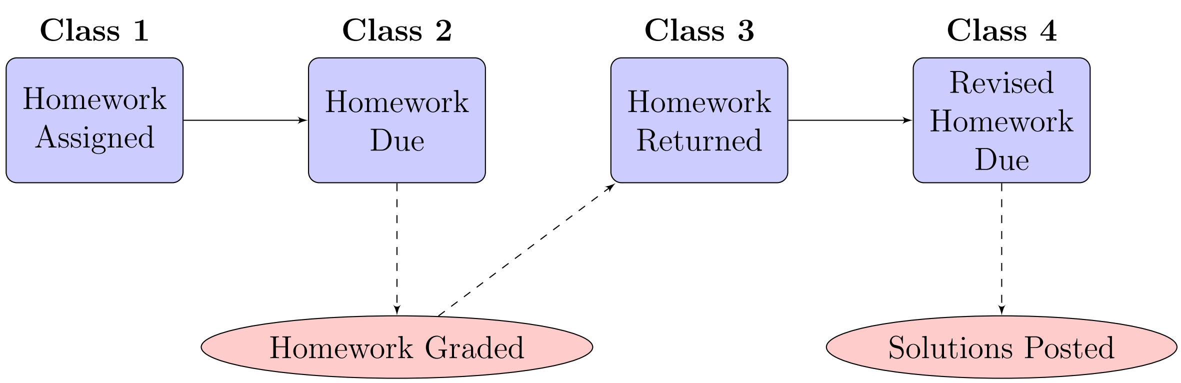
2. **Grading.** Half of your homework grade will come from your effort in attempting the problem, the remaining half will come from correctness (all or nothing).

3. **Turning In Work.** Homework is due at the end of the following class. If you are absent, you may email your homework by 9:55am of the due date. I prefer if you scan in your homework, but you can also email me a readable photo. NO LATE HOMEWORK WILL BE ACCEPTED.

4. **Recuperating Lost Points (Optional).** If you found you made a mistake on a homework assignment, you may redo a problem to recuperate your lost points. You must do this by the class following the one your homework was returned. It will be due at the end of class. If you failed to turn in homework, you are not eligible to do this.

5. **Solutions.** After all revised homework assignments are collected, I will post solutions for you to review.

Below is a diagram explaining the homework process from beginning to end.



1.2.2 Effort

The effort you put into this class will influence your grade. Some portion of your points will come directly from the effort you put in. When deciding your grade for effort, I will consider:

- **Attendance.** If you are chronically absent, I will take that into account when distributing final grades. Remember, university policy states:

  > A student whose irregular attendance causes him or her, in the judgment of the instructor, to become deficient scholastically, may run the risk of receiving a failing grade or receiving a lower grade than the student might have secured had the student been in regular attendance (Policy 42-27).

- **Class Participation.** Points are awarded based on your participation in class. It's sufficient to ask and answer questions.

1.2.3 Midterm

I have no control over the midterm. It will cover chapter 12 and chapter 13 of your textbook. Calculators are not permitted. The midterm is worth 100 points.

1.2.4 Final

I also have no control over the final. It will be cumulative, so it will cover chapters 12, 13, and 14 of your textbook. Calculators are not permitted. The final is worth 150 points.

1.3 What is Expected of You

Here are my rules for this class:

1. **If something is wrong, talk to me ASAP.** Sometimes major things will come up in your life which will force your education to be temporarily put aside. Talk to me as soon as you know (before you miss days of class and homeworks); I will work with you so you don't fall behind.

2. **Bring me ODS paperwork.** If this applies to you, please bring me your ODS paperwork as soon as you can. It is very important to me that all my students are properly accommodated for so they have the tools they need to excel. Please don't wait to do this.

3. **Don't cheat!** Give yourself the pleasure of earning the grade you deserve. I have never had a student cheat in my class; if you do the homework, you won't need to cheat to do well.

4. **Don't be afraid to ask for help.** You can always visit me in office hours or make an appointment. I also encourage you to work on the homework in groups. Talking through problems is something mathematicians do all the time and is a great way to learn the material. Don't forget to check out the free tutoring services at Penn State Learning!

5. **Be excited and have a good time.** There is so much we can understand with this stuff; it really opens up the world!

1.4 The General Approach to any Problem

The art of problem solving is multifaceted. If you give it some thought, it is actually a very creative process. Math can feels like a rigid discipline, but there are often many ways to solve a single problem.

When a challenging problem presents itself, I will ask you to use the following approach. I will use the acronym A METHOD to refer to this procedure.

1. **Absorb**: Spend time studying the problem. Read the problem carefully, think about what it's saying, and draw some kind of representation.

2. **Meditate**: Think; make sure to *understand* the problem. What is it asking?

3. **Estimate**: Estimate the answer. What kinds of answers would make sense in this context? This might be as vague, like "The answer is bigger than 30."

4. **Translate**: Write a mathematical statement describing everything.

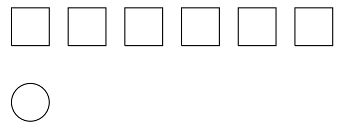
5. **Hone in On**: Use mathematical techniques to solve the "mathematized" problem.

6. **Deliberate**: Revisit the problem and compare it to the solution. Does the solution make sense? Does the technique fit the situation?

To understand how A METHOD works, let's look at an algebra problem.

Example 1.4.0.1 *Pearl has six times as many dimes as quarters in her piggy bank. She has 21 coins in her piggy bank totaling $2.55. How many of each type of coin does she have?*

1. **Absorb.** Let dimes be represented by squares and quarters be represented by circles. The problem tells us there are six dimes to every quarter in Pearl's piggy bank. Let's draw that out.

   She has to have 21 coins total and the value adds up to \$2.55.

2. **Meditate.** The problem gives us a ratio (6 to 1), a total number (21) and a total value(\$2.55). We can probably focus on the total number of coins and use our picture above (which has 7 coins) to solve it.

3. **Estimate.** The total value in our picture above is more than 50 cents, so we know we have less than five quarters.

4. **Translate.** Let x represent the number of groups of coins we drew above (6 dimes and 1 quarter). Then $7x = 21$

5. **Hone in On.** After dividing both sides by 7, we get $x = 3$. Therefore, we have 3 quarters and 18 dimes.

6. **Deliberate.** This makes sense because the total value of the coins needs to add up to \$2.55 and
$$(3)(.25) + (18)(.1) = .75 + 1.8 = 2.55.$$

Whenever you find yourself stuck on a problem, list the steps of A METHOD and try to see *where* you cannot move forward. This will often provide insight into what is confusing you. You'll know what to look up and, if that doesn't help, you'll have very specific questions you can ask anyone.

Always remember, things like

- getting stuck,
- feeling confused, and
- making a mistake

are good things if you take your time and understand what is going on. Studies have shown that when this happens, students learn the material *better* than if they got it right the first time. That extra effort makes your brain grow!

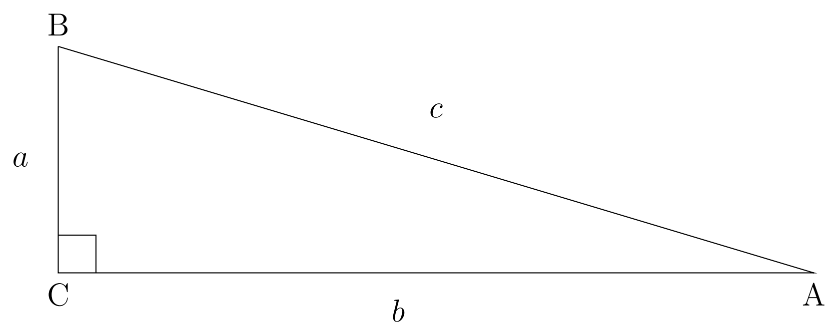
2 Preliminary Knowledge

Objectives

- ☐ I know topics from previous courses I will need for this course.
- ☐ I am familiar with the techniques of each of these topics:
  - ☐ Geometry
  - ☐ Trigonometry
  - ☐ Calculus
  - ☐ Three-Dimensional Drawing

We now review some topics from previous classes.

2.1 Geometry



The **pythagorean theorem** states that the sum of the squared lengths of the legs of a right triangle sum to the squared length of its hypotenuse.

$$a^2 + b^2 = c^2$$

From this theorem, we have a formula for the distance between two points.

$$c = d(A, B) = \sqrt{a^2 + b^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

By Δx and Δy, I mean "change" in the x direction and "change" in the y direction.

Example 2.1.0.2 *Determine the distance between $A(3,2)$ and $B(4,5)$*

We compute by looking at the changes in the x direction and the changes in the y direction.

$$d(A,B) = \sqrt{(4-3)^2 + (5-2)^2} = \sqrt{10}$$

2.2 Topics from Trigonometry

Trigonometry has a way of creeping into every calculus course you'll ever take; therefore, it is important to know the main identities. Here are a list of identities you are expected to know.

1. Negative angles in trigonometric functions
   - $\cos(-\theta) = \cos\theta$, (an **even** function)
   - $\sin(-\theta) = -\sin\theta$, (an **odd** function)
2. $\cos^2\theta + \sin^2\theta = 1$, which implies
   - $1 + \tan^2\theta = \sec^2\theta$
   - $\cot^2\theta + 1 = \csc^2\theta$
3. $2\sin\theta\cos\theta = \sin 2\theta$
4. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, which can also be written as
   - $\cos(2\theta) = 2\cos^2\theta - 1$
   - $\cos(2\theta) = 1 - 2\sin^2\theta$

2.2.1 Evaluating Trigonometric Functions

You need to know how to evaluate trigonometric functions over the standard angle values,

$$\theta = 0, \pi/6, \pi/4, \pi/3, \text{ and } \pi/2$$

These can be very difficult to memorize. I recommend you know the standard $30° - 60° - 90°$ and $45° - 45° - 90°$ triangles to deduce what they should be. Alternatively, if you're someone who prefers memorization, I recommend you memorize the squared values instead. They follow an easy pattern (see below).

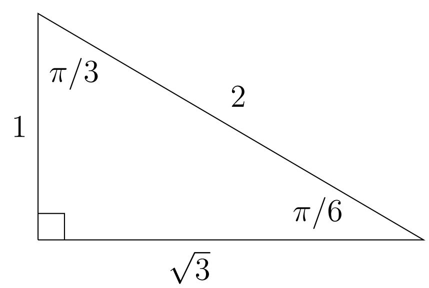
θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin^2\theta$	0/4	1/4	2/4	3/4	4/4
$\cos^2\theta$	4/4	3/4	2/4	1/4	0/4
$\tan^2\theta$	0/4	1/3	2/2	3/1	$4/0=\infty$

Example 2.2.1.1 *Find* $\sin(\pi/3)$.

If you have memorized the table, you will solve this the following way:

$$\sin\left(\frac{\pi}{3}\right)=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$$

If you want to use the $30^\circ - 60^\circ - 90^\circ$ triangle, you'll draw



Then, knowing that sine is opposite over hypotenuse, you will divide to get

$$\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

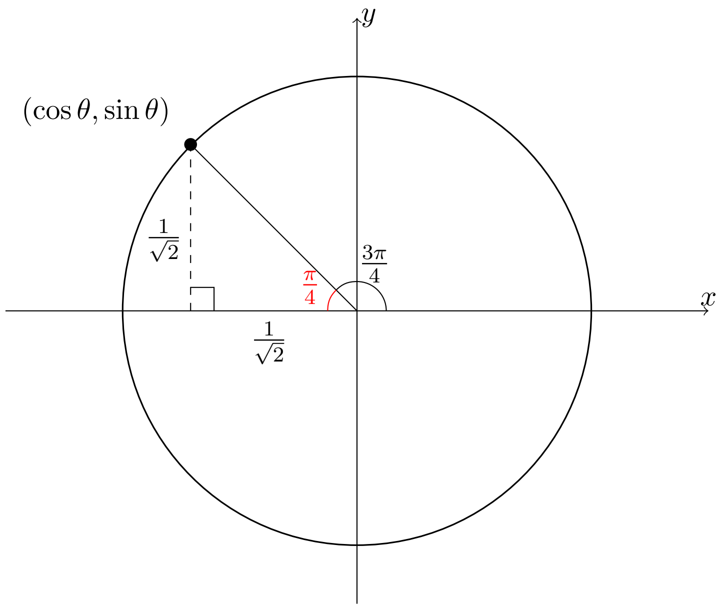
Use whichever method that best suits you.

2.2.2 The Unit Circle

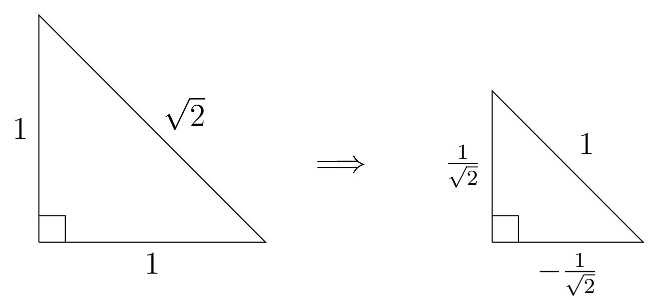
You should be familiar with the **unit circle**, the circle of radius *one* centered at the origin. It's a tool that allows you to evaluate trigonometric functions at certain higher angles, like $5\pi/4$.

Begin by plotting the angle on the unit circle. At the point where the line intersects the curve, draw a straight line down to the x-axis to create a right angle. Based on the triangle created, you can find $\cos\theta$, $\sin\theta$, and all other related trigonometric functions.

Example 2.2.2.1 *Find* $\sin 3\pi/4$ *and* $\cos 3\pi/4$.



We begin first by drawing a line in the circle that represents $3\pi/4$. From the point on the circle, we draw a straight line going down. We have drawn a triangle. One of its angles is $\pi/4$ (we know this because $\pi/4 + 3\pi/4 = \pi$, the total degrees of the top-half of the circle). We then recognize this triangle to be a $45° - 45° - 90°$ triangle. Rescale the triangle so the hypotenuse is one.



We then see that the x-value is $-\frac{1}{\sqrt{2}}$, so

$$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$

Similarly, the y-value is $\frac{1}{\sqrt{2}}$, so

$$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

2.3 Topics from Calculus

You are expected to know all the topics from Math 140 and Math 141 (i.e. the first and second courses of calculus). In our review, I want to place a special emphasis on parametric equations, basic derivatives, and basic integrals.

2.3.1 Parametric Equations

The unit circle turns out to be a **parametric equation**, an equation whose variables are dependent on a parameter. For the unit circle, $x = \cos\theta$ and $y = \sin\theta$. From the identity, $\sin^2\theta + \cos^2\theta = 1$, we get the equation $x^2 + y^2 = 1$. This is the equation of a circle. The parameter here is θ.

A more common parameter you'll see is time, t. For example,

$$(x, y) = (f(t), g(t))$$

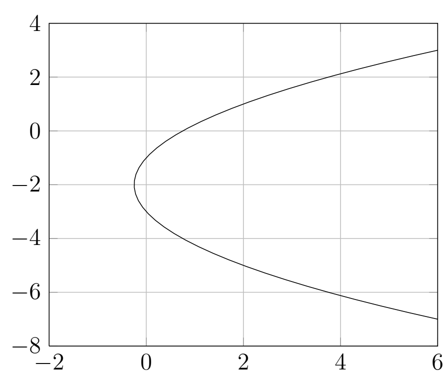
where $x = f(t)$ and $y = g(t)$. To graph a parametric equation, we write our a T-table for the values of t and find the corresponding ordered pairs.

Example 2.3.1.1 *Graph* $x = t^2 + t$ *and* $y = 2t - 1$.

To graph, we'd first write a T-table and then plot points.

t	$x = t^2 + t$	$y = 2t - 1$
-2	2	-5
-1	0	-3
0	0	-1
1	2	1

We use the variable t to represent some notion of time, so if we think of time as starting from $t = -2$ and ending at $t = 1$, then we'd graph just that portion.



2.3.2 Derivatives and Integrals

I'm not going to go too much into these topics, but you are expected to be familiar with the following concepts from calculus. If you aren't completely comfortable with any of the topics listed below, please talk to me. I can recommend some resources and work with you to get up to speed.

1. **Limits**

   - L'Hôpital's Rule

     $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

     when

     $$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x) \text{ are both } \infty \text{ or zero.}$$

2. **Derivatives**

   - Trig Function & Inverse Trig Function Derivatives

$f(t)$	$\frac{d}{dt}f(t)$
$\sin t$	$\cos t$
$\cos t$	$-\sin t$
$\tan t$	$\sec^2 t$
$\cot t$	$-\csc^2 t$
$\sec t$	$\sec t \tan t$
$\csc t$	$-\csc t \cot t$

$f(t)$	$\frac{d}{dt}f(t)$
$\sin^{-1} t$	$\frac{1}{\sqrt{1-t^2}}$
$\cos^{-1} t$	$-\frac{1}{\sqrt{1-t^2}}$
$\tan^{-1} t$	$\frac{1}{1+t^2}$
$\cot^{-1} t$	$-\frac{1}{1+t^2}$

- Product Rule

$$(fg)' = f'g + fg'$$

- Chain Rule

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

- Quotient Rule

$$\frac{d}{dx}\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

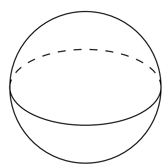
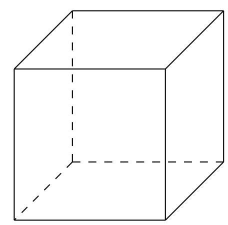
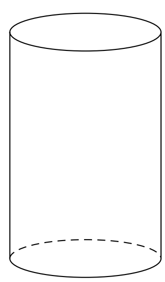
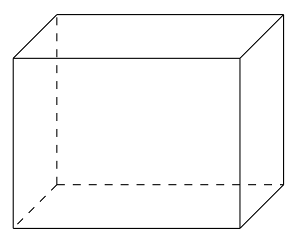
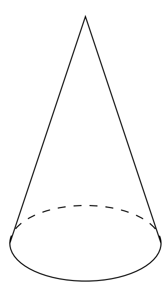
3. **Integrals**

- Integration by Parts

$$\int udv = uv - \int vdu$$

2.4 Drawing in Three Dimensions

When we draw things in three dimension on a two-dimensional surface, the image can be very confusing very fast. It is important that we explore the traditional ways of representing these objects. Basically, we draw added lines to show depth and we draw them dashed when they would normally be obscured to the viewer. Below are six examples of three dimensional objects represented on a two dimensional surface.

Shape (Circles)	Picture	Shape (Squares)	Picture
Sphere		Cube	
Cylinder		Rectangular Prism	
Cone		Triangular Prism