

3.5 Equations of Lines and Planes

Objectives

- I know how to define a line in three-dimensional space.
- I can write a line as a parametric equation, a symmetric equation, and a vector equation.
- I can define a plane in three-dimensional space and write an equation to represent it.
- I can calculate the distance between a point and a plane.

Up until now, we've graphed points, simple planes, and spheres. In the next two sections, we will explore other types of equations.

Lines

Let's begin with lines. What defines a line? Your answer might be one of the following

- two points
- a point and slope

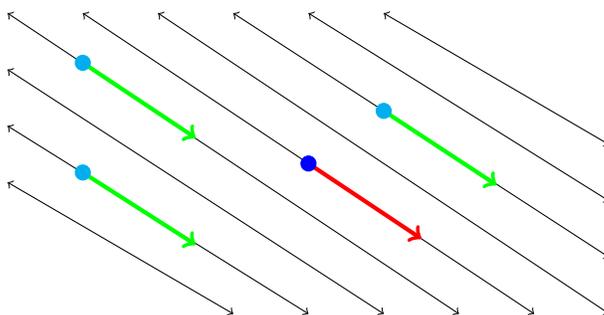
In three dimensions, the answer is the same; however, we have the notion of a vector at our disposal. A vector is an object of movement (just like slope) and it can be constructed from two points. Hence, a third and equivalent definition of a line is

- a point and a vector

Why do we need a point? Remember that a vector has no location. It only describes a motion, but it doesn't give you a starting point. In other words, if you're given a vector



you can define infinitely many lines that travel in the direction of your vector.



Now that we know what defines a line, how do we write the equation? It turns out there are three ways:

- We write a **parametric equation**. Given a vector $\vec{v} = \langle 4, -2, 2 \rangle$ and a point $P(-3, -1, 0)$, the parametric equation is

$$x(t) = -3 + 4t, \quad y(t) = -1 - 2t, \quad z(t) = 0 + 2t$$

- Based on the method above, we can set each equation equal to t and equate all three equations. This gives us a **symmetric equation**

$$\frac{x - (-3)}{4} = \frac{y - (-1)}{-2} = \frac{z - 0}{2}$$

Note that you cannot use this method if any entries of the vector are zero.

- The third method is to write the formula as a **vector equation**, which we will explore more in-depth in the next chapter.

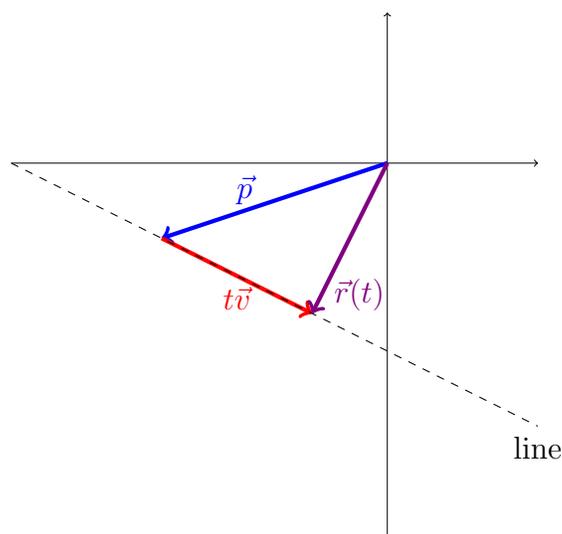
$$\vec{r}(t) = \langle -3, -1, 0 \rangle + t\langle 4, -2, 2 \rangle$$

In all three instances, we treat the vector as the slope and the point as the intercept.

Before we move onto the examples, let's take a moment to think about the vector equation formula. Notice that we turned a point into a vector. That is, $P(-3, -1, 0)$ is a point while $\vec{p} = \langle -3, -1, 0 \rangle$ is a vector. This seems really weird compared to the other two equations.

What distinguishes a vector equation from the others is that it takes in a scalar value, t , and produces a vector, $\vec{r}(t)$. The other two equations produce points.

What is this vector? Well, let's take a moment to think about it. It is the sum of two vectors: one that points from the origin to the line and another that points along the line multiplied by t .



Remember that in real life, the origin is determined by convenience. We might be thinking of how one particle, like a comet, moves relative to another, like the earth. We would think of the earth as the origin. If we wanted to shoot the comet with lasers, we'd use a vector equation to help us describe where to aim our laser at time t .

Straight lines are not the only shape vector functions can take. We will see more varieties of vector functions in the next chapter.

3.5.1 Examples

Example 3.5.1.1 Define the line that connects the points $(1, 4, 7)$ and $(2, 6, 5)$.

First, we need to describe the motion of the line. We do this by constructing the vector moving from one point to another.

$$\vec{v} = \langle 2 - 1, 6 - 4, 5 - 7 \rangle = \langle 1, 2, -2 \rangle$$

We pick one of the points. It doesn't matter which one. Let's pick $(1, 4, 7)$. This is the point where the line "starts" when $t = 0$. Now that we have a point and a vector, we have all that we need to define the line. We can express it in three ways.

1. the parametric equations

$$x(t) = 1 + t$$

$$y(t) = 4 + 2t$$

$$z(t) = 7 - 2t$$

2. the symmetric equation

$$\frac{x-1}{1} = \frac{y-4}{2} = \frac{z-7}{-2}$$

3. the vector equation

$$\vec{r}(t) = \langle 1, 4, 7 \rangle + t \langle 1, 2, -2 \rangle = \langle 1 + 2t, 4 + 2t, 7 - 2t \rangle$$

There are many different ways to do this problem. We could have picked the vector to point in the opposite direction or we could have picked the other point or both. The definition of an infinite line isn't unique.

Example 3.5.1.2 Define the line segment that begins at $(3, 2, 9)$ and ends at $(3, 4, 2)$.

Here, the orientation matters a great deal. We want to start at $(3, 2, 9)$ and end at $(3, 4, 2)$. It's important that our vector points in the right direction. That is

$$\vec{v} = \langle 3 - 3, 4 - 2, 2 - 9 \rangle = \langle 0, 2, -7 \rangle$$

We will use the starting point $(3, 2, 9)$ as our point. Now, let's list our definitions.

1. The parametric equations are

$$\begin{aligned} x(t) &= 3 \\ y(t) &= 2 + 2t \\ z(t) &= 9 - 7t \end{aligned} \quad 0 \leq t \leq 1$$

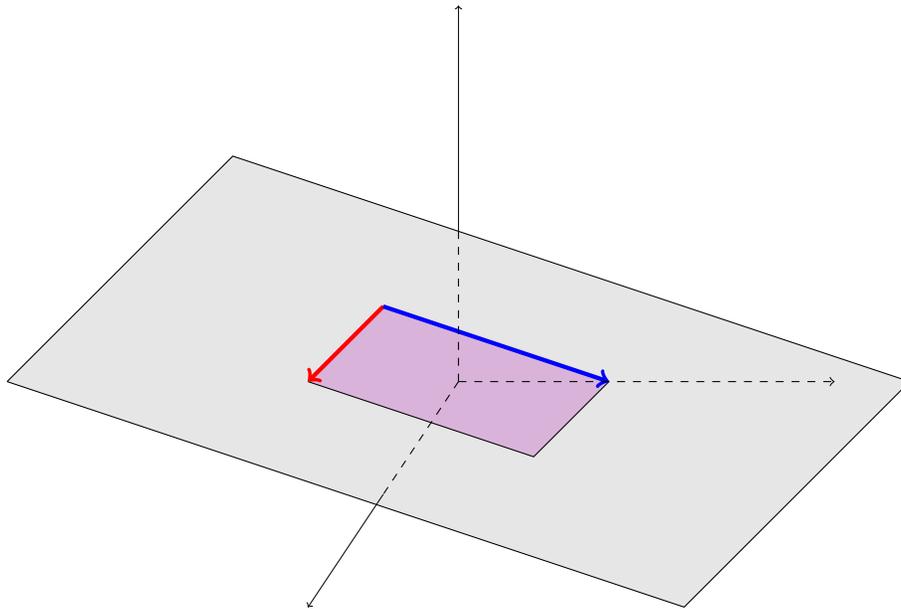
At $t = 0$, our point is $(3, 2, 9)$. At $t = 1$, our point is $(3, 4, 2)$. So our line segment as the restriction on t .

2. We cannot do a symmetric equation because there is no way to determine where to start and stop.

3. The vector equation is

$$\vec{r}(t) = \langle 3, 2, 9 \rangle + t \langle 0, 2, -7 \rangle = \langle 3, 2 + 2t, 9 - 7t \rangle \quad 0 \leq t \leq 1$$

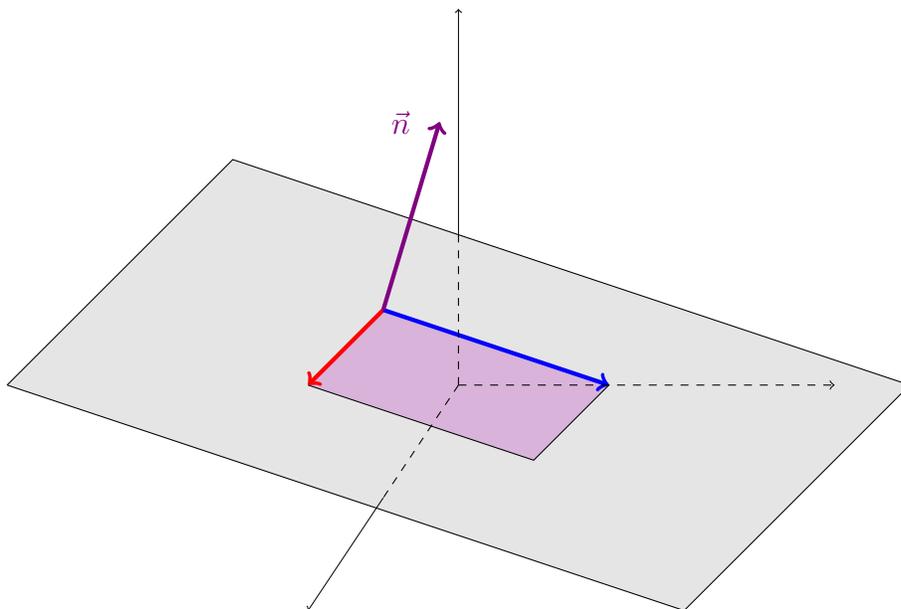
Now, let's consider a plane. Think back to our parallelograms. If we allow them to grow to infinity, we have a plane.



This parallelogram is defined by two vectors traveling in two distinct directions. We would still need a point to give the object location.

So, theoretically, we can define a plane with two vectors and a point. But can we do better?

Yes! We can look at the cross product of the two vectors instead!



We call this vector the **normal vector** because it is *normal* to the plane. This is a good time to note all the terms and phrases that mean the same thing:

- normal
- perpendicular
- orthogonal
- forms a right angle with

Why is it better to use this perpendicular vector than the two vectors inside the plane? It's true that less is better in math; however, the real reason is deeper than that. Normal vectors will correspond to *total derivatives* expressed as vectors. This will be a *very* important idea later on in this class. Be sure to keep an eye out for it.

For the normal vector

$$\vec{n} = \langle n_1, n_2, n_3 \rangle$$

and the point (a, b, c) , the equation that describes the vector is

$$n_1(x - a) + n_2(y - b) + n_3(z - c) = 0$$

You are expected to multiply it out and simplify on an exam.

3.5.2 Examples

Example 3.5.2.1 Define the plane perpendicular to the vector $\langle 2, -1, 4 \rangle$ that goes through the point $(1, 0, -1)$

We plug in the vector and point into our formula to get

$$2(x - 1) - (y - 0) + 4(z + 1) = 0$$

On an exam, this form will not be good enough. One has to multiply everything out

$$2x - 2 - y + 4z + 4 = 0$$

and place all the constants on the opposite side of the equals sign to the variables

$$\boxed{2x - y + 4z = -2}$$

Example 3.5.2.2 Define the plane perpendicular to the line

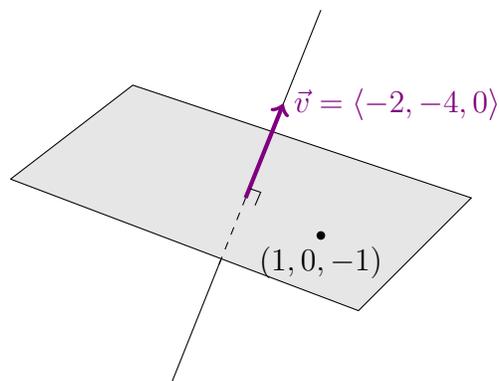
$$x(t) = 4 - 2t$$

$$y(t) = -1 - 4t$$

$$z(t) = 12$$

that goes through the point $(1, 0, -1)$

It is crucial to draw a picture in order to understand this problem.



Notice that \vec{v} , which is taken from the slopes of the parametric equations, will give us the vector that moves in the direction of the line. Since the line is perpendicular to the plane, then so is the vector. Therefore, our normal vector is the same vector! That is,

$$\vec{n} = \langle -2, -4, 0 \rangle$$

Now we use the formula

$$-2(x - 1) - 4(y - 0) + 0(z + 1) = 0$$

We multiply out to get

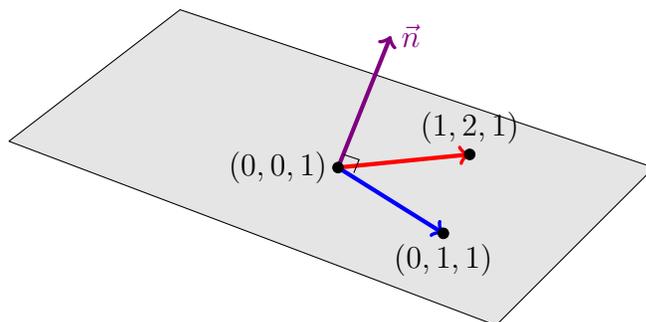
$$-2x + 2 - 4y = 0$$

and we rearrange. Our final answer is

$$\boxed{2x + 4y = 2}.$$

Example 3.5.2.3 Define the plane that contains the points $A(1, 2, 1)$, $B(0, 0, 1)$ and $C(0, 1, 1)$.

To define a plane, we need a normal vector and a point. Remember that we can construct a normal vector from two distinct vectors inside the plane. Let's draw a picture.



With the three points in the plane, we can construct two vectors, take the cross product, and find the normal vector. In our picture, we picked $(0, 0, 1)$ to be the point in common between the two vectors, but we could have picked any one of those points to be that point. Draw a few pictures to prove it to yourself.

$$v_1 = \langle 1 - 0, 2 - 0, 1 - 1 \rangle = \langle 1, 2, 0 \rangle$$

$$v_2 = \langle 0 - 0, 1 - 0, 1 - 1 \rangle = \langle 0, 1, 0 \rangle$$

Then

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \langle 1 - 0, -(0 - 0), 1 - 0 \rangle = \langle 1, 0, 1 \rangle$$

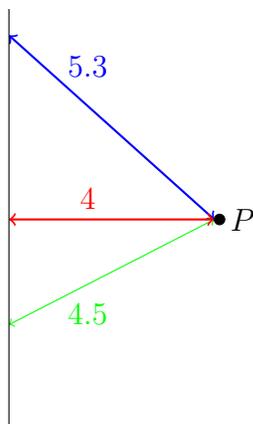
We pick any one of the points to get a definition for the plane. Let's pick $(0, 0, 1)$ since it has a lot of zeros.

$$1(x - 0) + 0(y - 0) + 1(z - 1) = 0$$

This simplifies to

$$\boxed{x + z = 1}$$

In this last portion, we want to give some thought about the distances between objects. Let's first think of a line and a point P .

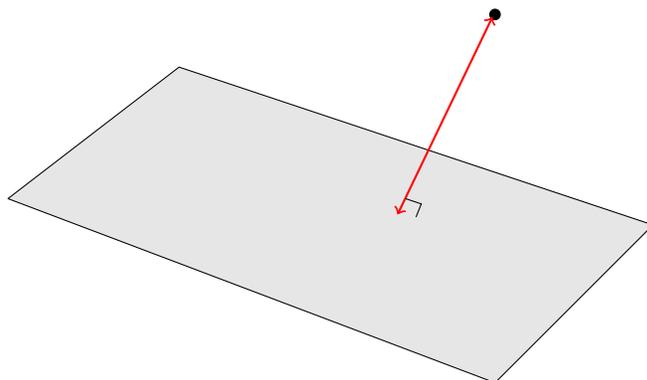


We could draw many lines between P and the line. Which one is the distance between the two objects? Well, in reality, we're only interested in the shortest path. So in the above example, the distance is 4.

What do you notice about the shortest path from the point to the line? How is the path oriented relative to the line? Take a moment to think about this.

If you noticed that the path and the line are perpendicular (or orthogonal or normal), then you noticed the right feature. The shortest path will always have this feature.

Now let's extend this to the distance between a point and a plane.



We know that the path must be perpendicular to the plane, but how do we find this path?

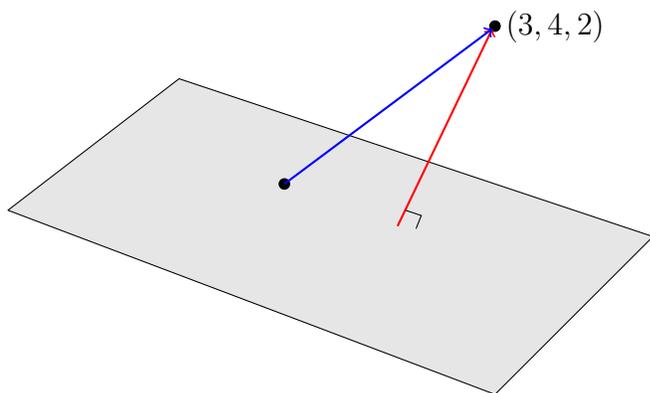
If the equation of the equation of the plane is

$$2x + y + 3z - 10 = 0$$

then the vector $\langle 2, 1, 3 \rangle$ is perpendicular to the plane. Is that the vector we want?

No, it turns out that it's not the appropriate vector because while it moves in the right direction, it doesn't travel the right amount of distance. The vector we need must describe a movement that is perpendicular to the plane and just from the plane to the point. How can we find this vector?

We find it using projections! We can define a vector that goes from the plane to the point and project it onto the vector that is normal to the plane.



Let's define that vector. First, we must find a point on the plane. If $x = 1$ and $z = 2$, then $y = 2$ since $2(1) + (2) + 3(2) = 10$. So our point is $(1, 2, 2)$.

The vector we make is

$$\vec{v} = \langle 3 - 1, 4 - 2, 2 - 2 \rangle = \langle 2, 2, 0 \rangle$$

Now, we project this vector onto the normal vector.

$$\vec{v} = \frac{\langle 2, 2, 0 \rangle \cdot \langle 2, 1, 3 \rangle}{|\langle 2, 1, 3 \rangle|^2} \langle 2, 1, 3 \rangle = \frac{3}{7} \langle 2, 1, 3 \rangle = \left\langle \frac{6}{7}, \frac{3}{7}, \frac{9}{7} \right\rangle$$

So the distance is

$$\frac{\sqrt{36 + 9 + 81}}{7} = \frac{\sqrt{126}}{7} \approx 1.6$$

It's important to know how this process works in order to figure out the geometry for a given situation. Still, it may be helpful to know the formula for determining this distance. Given a point $(\hat{x}, \hat{y}, \hat{z})$ to a plane $ax + by + cz + d = 0$ is

$$D = \frac{|a\hat{x} + b\hat{y} + c\hat{z} + d|}{\sqrt{a^2 + b^2 + c^2}}$$

3.5.3 Examples

Example 3.5.3.1 Find the distance between the point $(3, 0, -1)$ and the plane $3x + y + z = 13$

We can use the formula:

$$D = \frac{|3(3) + (0) + (-1) - 13|}{\sqrt{3^2 + 1^2 + 1^2}} = \frac{5}{\sqrt{11}} \approx 1.51$$

Summary of Ideas: Equations of Lines and Planes

- Lines are defined by two points or, equivalently, a point (p, q, r) and a vector parallel to its direction $\langle v_1, v_2, v_3 \rangle$.

- We can describe a line in three dimensions one of three ways.

- A set of parametric equations, which sends a scalar t to a point $(x(t), y(t), z(t))$. Its formula is

$$x(t) = p + v_1 t$$

$$y(t) = q + v_2 t$$

$$z(t) = r + v_3 t$$

- A symmetric equation, which gives us sets of points (x, y, z) . Its formula is

$$\frac{x - p}{v_1} = \frac{y - q}{v_2} = \frac{z - r}{v_3}$$

Note that you cannot use this formula if any of the entries of the vector are zero.

- A vector equation, which sends a scalar t to a vector $\langle x(t), y(t), z(t) \rangle$. Its formula is

$$\vec{r}(t) = \langle p, q, r \rangle + t \langle v_1, v_2, v_3 \rangle$$

- A plane is defined by a point (p, q, r) and a vector normal to it $\langle n_1, n_2, n_3 \rangle$.

- Its formula is

$$n_1(x - p) + n_2(y - q) + n_3(z - r) = 0$$

- You can define a plane with different objects, like three points, but from them, you need to find a normal vector and a point.

- The distance from a point $(\hat{x}, \hat{y}, \hat{z})$ to a plane $ax + by + cz + d = 0$ is

$$D = \frac{|a\hat{x} + b\hat{y} + c\hat{z} + d|}{\sqrt{a^2 + b^2 + c^2}}$$