5.2 Limits and Continuity

Objectives

- ☐ When the limit point is in the domain, I know how to calculate the limit.
- ☐ If I can simplify the expression, I know how to calculate the limit.
- ☐ If I suspect there is no limit, I know how to find two distinct limits.
- ☐ I know how to determine if a function is continuous.

Limits are tricky when looking at multivariate functions. Why? Before, we looked at limits as a single variable went to a number, like $x \to 4$. Now our limits consider an ordered pair going to an ordered pair, like $(x, y) \to (4, 5)$. The catch is that ordered pairs can move along any path (it doesn't even have to be a line!). In order for a limit to be legitimate, it has to work on *any* path we take.

To determine a limit of a multivariable function, we will consider the following algorithm.

I. Is the limit point in the domain of the function?

   YES: Plug in the point. The solution is the limit.

   NO: Continue onto the next step.

II. Can you simplify the expression?

   YES: Simplify and return to step I.

   NO: Continue onto the next step.

III. If steps I and II have failed, you should *suspect* that there is no limit. In reality, there may still be a limit, but it will be a difficult one to find. For exercises in this class, there likely won't be a limit. We will prove that the limit does not exist by computing the limit along two different paths that result in different solutions.

Let's see how to use the algorithm in practice with the following examples.

5.2.1 Examples

Example 5.2.1.1 *Does the limit exist? If so, compute it. If not, prove it.*

$$\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$$

First, observe that $(0,0)$ is not in the domain of $\frac{x^4-4y^2}{x^2+2y^2}$. So we answer NO for Step I and move onto Step II.

Notice that we *can* simplify this expression. We get

$$\frac{x^4-4y^2}{x^2+2y^2} = \frac{(x^2-2y^2)(x^2+2y^2)}{x^2+2y^2} = x^2-2y^2.$$

Now we return to Step I and observe that $(0,0)$ is in the domain of x^2-2y^2. Therefore, we can plug in.

Thus,

$$\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2} = (0)^2-2(0)^2 = \boxed{0}$$

Example 5.2.1.2 *Does the limit exist? If so, compute it. If not, prove it.*

$$\lim_{(x,y)\to(1,-1)} e^{-xy}\cos(x+y)$$

Notice that the point (1, -1) is in the domain of the function. By Step I, we can plug in to find the limit:

$$e^{-(1)(-1)}\cos(1-1) = e\cos(0) = e$$

Therefore, we limit is

$$\lim_{(x,y)\to(1,-1)} e^{-xy}\cos(x+y) = \boxed{e}$$

Example 5.2.1.3 *Does the limit exist? If so, compute it. If not, prove it.*

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+y^2}$$

By our algorithm, we should first make two observations. The first is that $(1,0)$ is not in the domain of the function (when we plug in, we get $0/0$). Also, we cannot simplify this function since the denominator can't be factored. Therefore, we should suspect that there is no limit.

Finding two different limits is a bit of an art form.

For the first limit, let's pick the path $x = 1$. That means $x = 1$ and $y \to 0$. So it becomes a one-dimensional limit. That is,

$$\lim_{y\to 0} \frac{y-y}{(1-1)^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

The second path, let's consider $y = x-1$ and we let $x \to 1$. Along this path, we get

$$\lim_{x\to 1} \frac{x(x-1)-(x-1)}{(x-1)^2+(x-1)^2} = \lim_{x\to 1} \frac{(x-1)^2}{2(x-1)^2} = \frac{1}{2}$$

Our solution is: $\boxed{\text{Since } 1/2 \text{ and } 0 \text{ are two distinct numbers, the limit does not exist.}}$.

How was I able to find the second path? I picked what I like to call a "smart path." A smart path relates the numerator and denominator in an expression to give a nonzero limit. In this case, I thought

$$xy - y \sim (x-1)^2 + y^2 \quad \Longrightarrow \quad y(x-1) \sim (x-1)^2 + y^2$$

Looking at that second expression, I can see that if $y = (x-1)$, then the expressions on both sides are similar. One is $(x-1)^2$ and the other is $2(x-1)^2$. We need to study a few more examples to help us see how to find smart paths.

Example 5.2.1.4 *Does the limit exist? If so, compute it. If not, prove it.*

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Here, we again suspect the limit does not exist. We need to find two different limits.

The first limit is just a simple path, like $x = 0$. Along this path, we assume $x = 0$ and see what happens as $y \to 0$. That is,

$$\lim_{y\to 0} \frac{0}{0^2+y^2} = 0$$

The second limit should come from a "smart" path. We want to consider a path that will make xy similar to $x^2 + y^2$. What is we pick $y = x$? In this case, the numerator becomes x^2 and the denominator becomes $2x^2$. So our limit is $1/2$. That is,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Limits are used to understand continuity. A function $f(x, y)$ is called **continuous at** (a, b) if the limit exists, i.e.

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We may ask where a function is continuous. The answer is simply all the points *inside* the domain.

5.2.2 Examples

Example 5.2.2.1 *Determine the set of points at which the function is continuous.*

$$f(x, y) = \frac{xy}{1 + e^{x-y}}$$

Notice that $1+e^{x-y}$ will always be nonzero. So all possible points will be in the domain and have a limit. Therefore, the function $f(x, y)$ is continuous everywhere. We write $\boxed{D = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}.}$

Example 5.2.2.2 *Determine the set of points at which the function is continuous.*

$$f(x, y) = \frac{e^x + e^y}{e^{xy} - 1}$$

Here, the denominator is $e^{xy} - 1$, which can equal zero if $x = 0$ or $y = 0$. So our domain excludes those possibilities. Therefore, $\boxed{D = \{(x, y) | x \neq 0 \text{ or } y \neq 0\}}$. If we were to graph this, we'd exclude the axes.

Example 5.2.2.3 *Determine the set of points at which the function is continuous.*

$$f(x, y) = \frac{1}{\sqrt{y^2 - 1}}$$

This function exists so long as $y > 1$. Therefore, the domain is $\boxed{D = \{(x, y) | y > 1\}}$

Summary of Ideas: Limits and Continuity

- If a limit exists, we can often (though not always) simplify the expression and plug in.
- If the limit does not exist, we prove this by finding two different limits. One of the paths we pick, the "smart path," relates the numerator and denominator. Put another way, it makes the numerator and denominator differ by a constant.
- A function is continuous at a point (a, b) if

$$\lim_{(0,0)\to(a,b)} f(x, y) = f(a, b)$$