5 Chapter 14: Partial Derivatives

In the previous chapter, we studied vector functions

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

which took in a scalar t and spit out a vector $\vec{r}(t)$. In this chapter, we will study functions that take in multiple scalar inputs, like x and y, but produce just one scalar output

$$z = f(x, y).$$

These are called *functions of several variables.* They are the main object of study in multivariate calculus.

5.1 Functions of Several Variables

Objectives

- ☐ I know how to find the domain of a function of several variables. If there are two or three input variables, I know how to graph the domain.
- ☐ I know how to find the range of a function of several variables. I can graph it.
- ☐ I know how to graph level curves when there are two input variables.

The first step in understanding any function is being able to recognize its **domain** and **range**.

Definition 5.1.1 *The* **domain***, D, of a function of many variables, like $f(x, y)$, is the set of values f takes in. It's* **range***, R, is the set of values f spits out.*

While this sounds simple, in practice we have to consider situations we didn't in two-dimensional calculus. Namely, we have to think about the variables individually *and* together.

Here are some things to keep in mind when you are doing this problems:

- The range will always be one-dimensional.
- The domain will have the same number of dimensions as there are input variables.
- The domain needs to include *all* possible inputs, which can make graphing it very tricky.

5.1.1 Examples

Example 5.1.1.1 *For the function below, find and sketch the domain then find its range.*

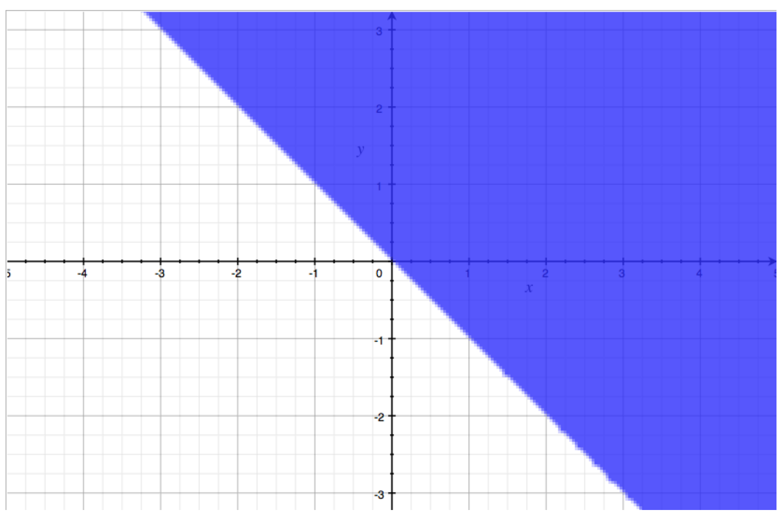
$$f(x, y) = \sqrt{x + y}$$

Any value under the square root must be greater than or equal to zero. Therefore, the domain is

$$D = \{(x, y) \mid x + y \geq 0\}$$

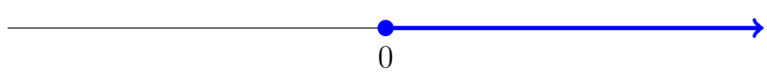
Surely if both x and y are positive numbers, then $x+y \geq 0$. But, is that enough? What if $x = 4$ and $y = -2$. Their sum is then 2, which is greater than 0. How to we account for cases like this?

This might not be obvious. To help us figure this out, we'll think about the extreme cases. That would be $x = -y$ and $y = -x$. These are cases like $(x, y) = (-4, 4)$ or $(x, y) = (2, -2)$. Both of these cases correspond to the same line: $y = -x$. Everything above this line will satisfy the domain restriction. Check a few points to convince yourself.



The range are all the values $\sqrt{x + y}$ can produce. These are all positive numbers. Therefore the range is

$$R = \{z \mid z \geq 0\}$$



Example 5.1.1.2 *For the function below, find and sketch the domain then find its range.*

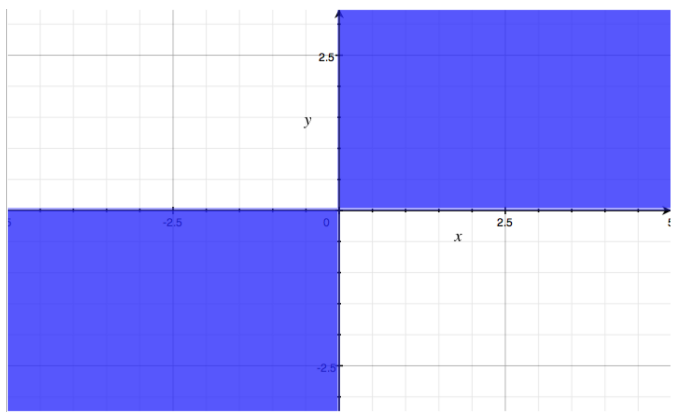
$$f(x, y) = \sqrt{xy}$$

As before, any value under the square root must be greater than or equal to zero. As a result, our domain is

$$D = \{(x, y) \mid xy \geq 0\}$$

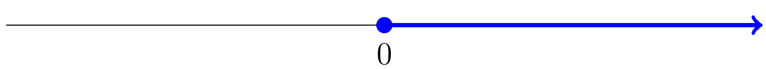
How do we graph this? If $xy \geq 0$, then we're looking at two regions.

1. $x \geq 0$ and $y \geq 0$
2. $x \leq 0$ and $y \leq 0$ (remember, the product of two negative numbers is positive!)



The range are all the values $\sqrt{xy}$ spits out. It can only be number greater than or equal to zero. So we have

$$R = \{z \mid z \geq 0\}$$



Example 5.1.1.3 *For the function below, find its domain and graph it.*

$$f(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2)$$

Then, find its range.

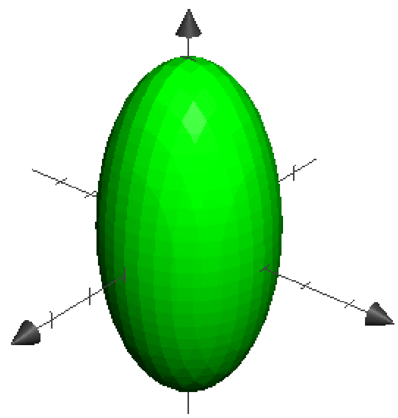
Values inside ln() must be strictly positive, so the domain is

$$D = \{(x, y, z) \mid 16 - 4x^2 - 4y^2 - z^2 > 0\}.$$

But is this really the best way to express this set? It doesn't tell us anything in this form. So, let's try to simplify the expression a little.

$$16 - 4x^2 - 4y^2 - z^2 > 0$$
$$16 > 4x^2 + 4y^2 + z^2$$
$$4 > x^2 + y^2 + \frac{z^2}{4}$$

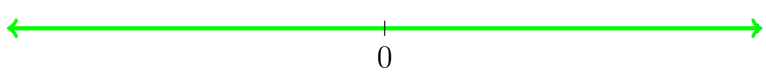
The graph of the domain is then a solid ellipse but missing its shell.



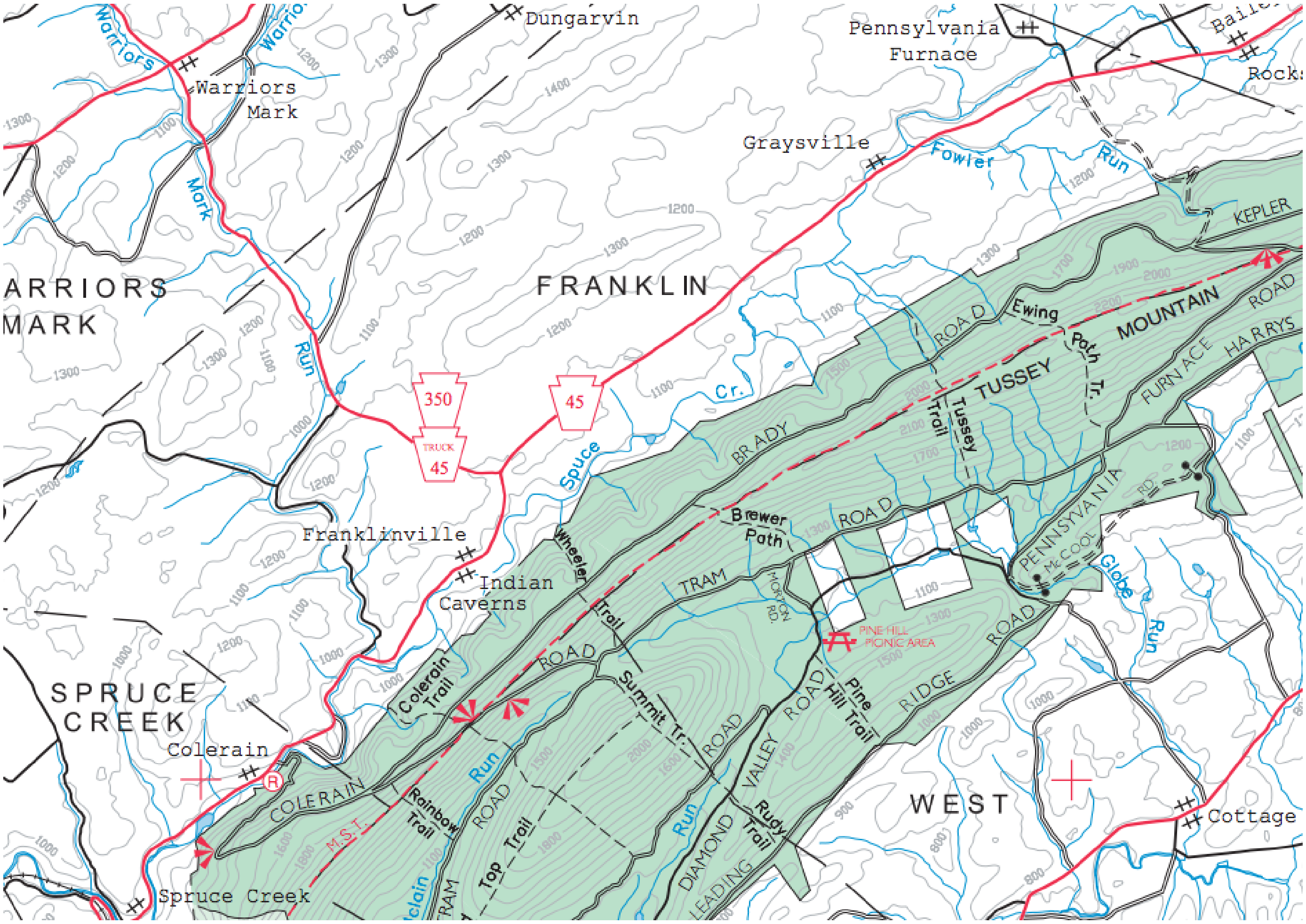
The natural log (ln) takes in only positive values, but produces all real numbers. Remember that the natural log of numbers between 0 and 1 are negative. So the range is

$$R = \{z \mid z \in \mathbb{R}\} = \{z \mid \text{ all real numbers}\}$$

Either expression is an acceptable answer.
Here is the graph of the range.



In addition to looking at the domain and range, we want to be able to graph functions in three dimensions. This is quite tricky. One popular method is to graph **level curves**. These are the lines one sees on a *topographic map*. Below is an example of a topographic map of Tussey Mountain.



Notice the lines looping around with jagged edges. Around the word "FRANKLIN," we see a loop labeled 1300. This tells us that on that line, the elevation is 1300ft.

Level curves are a way to think about f as a height. What we'll do is pick values for f and graph the resulting two-dimensional curve. This process won't be useful if you have more than two input variables.

5.1.2 Examples

Example 5.1.2.1 *Graph the level curves of*

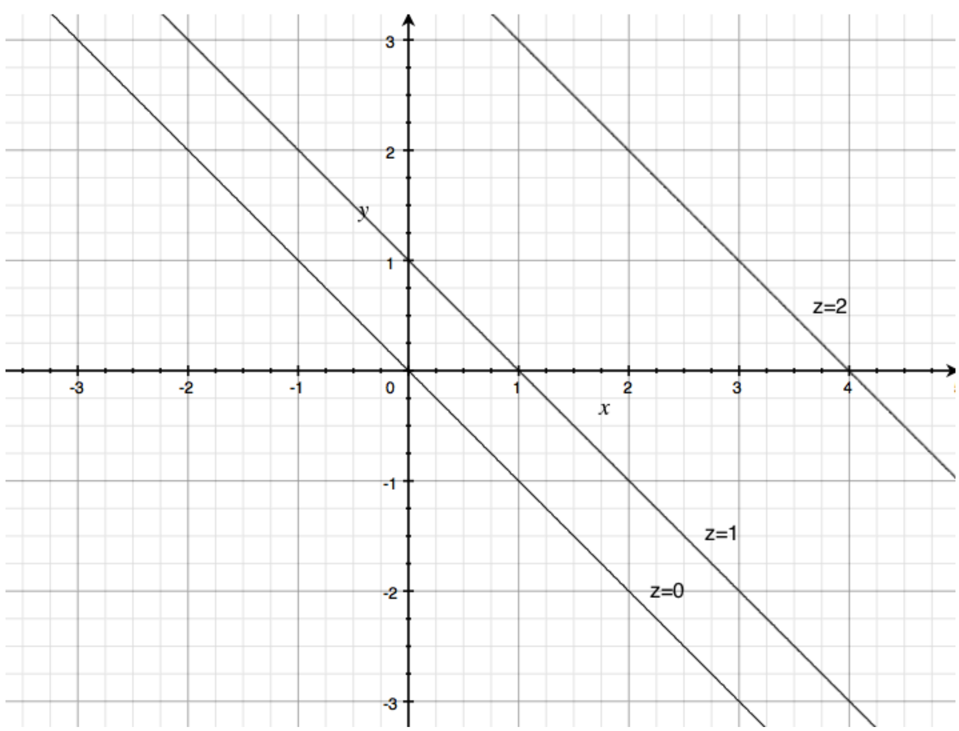
$$f(x, y) = \sqrt{x + y}$$

Use that information to sketch the 3 dimensional graph.

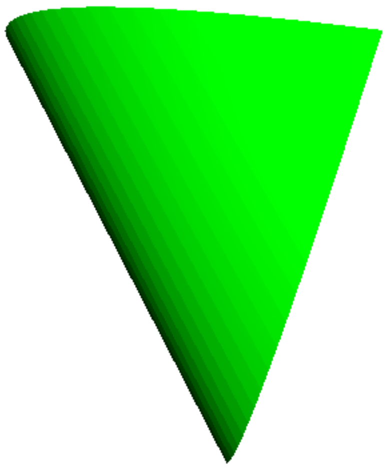
Let's pick values for f and write the respective functions.

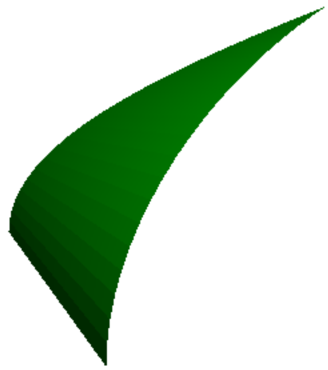
f	Function
0	$0 = \sqrt{x+y} \implies 0 = x + y$
1	$1 = \sqrt{x+y} \implies 1 = x + y$
2	$2 = \sqrt{x+y} \implies 4 = x + y$
3	$3 = \sqrt{x+y} \implies 9 = x + y$

Now, we graph all these lines on the same graph.



From this, we see that the curve is increasing in height but the increasing is slowing down. Here is the graph of the three dimensional surface seen from two different angles.





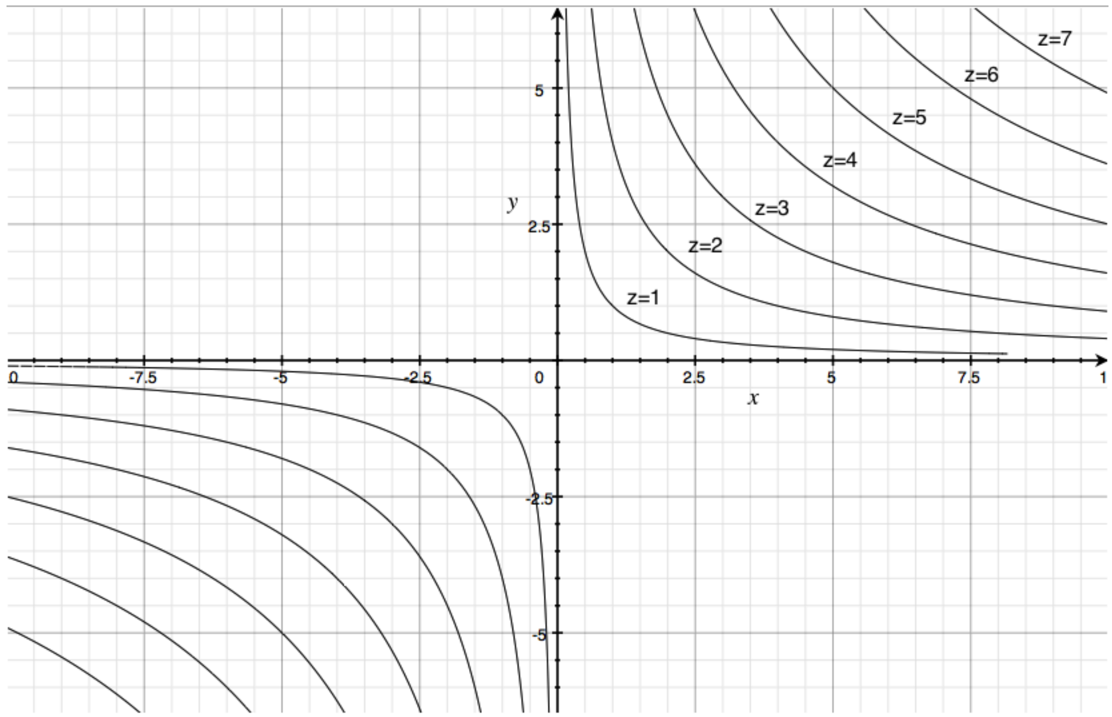
Example 5.1.2.2 *Graph the level curves of*

$$f(x, y) = \sqrt{xy}$$

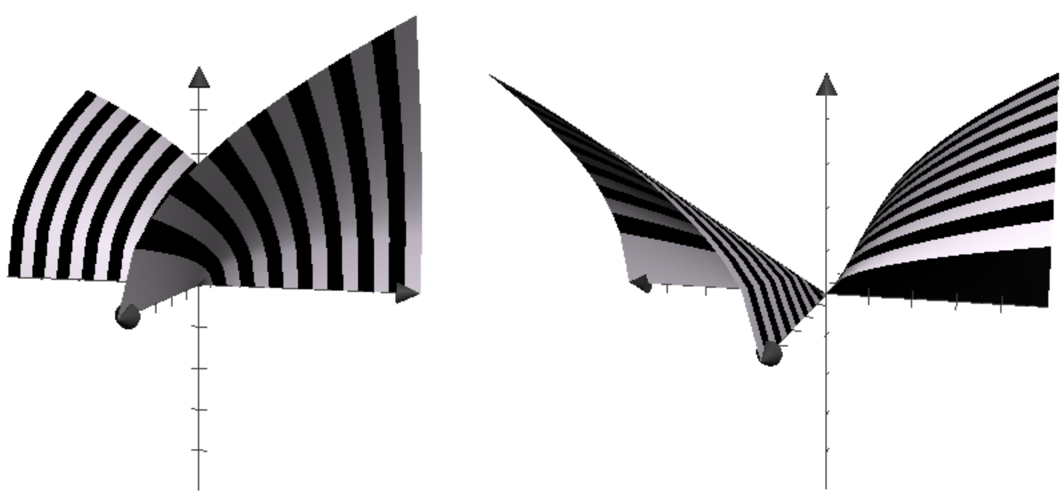
Use that information to sketch the 3 dimensional graph.

First, we pick z values. In the picture below, I have graphed $z = 1$, $z = 2$, ..., and $z = 7$.

f	Function
0	$0 = \sqrt{xy}$
1	$1 = \sqrt{xy}$
2	$2 = \sqrt{xy}$
3	$3 = \sqrt{xy}$
4	$4 = \sqrt{xy}$
5	$5 = \sqrt{xy}$
6	$6 = \sqrt{xy}$



Given the shape of the level curve, our graph is then



Summary of Ideas: Functions of Several Variables

- The **domain** is a set of inputs that are valid for the function.
- The **range** is all the values produced by the function. It will always be one-dimensional for functions of multiple variables.
- The **level curves** are the lines for various values of the function, f.
- Drawing level curves is a technique for graphing three-dimensional surfaces.