## 4.2 Derivatives and Integrals of Vector Functions

**Objectives**

- ☐ I can take the derivative of a vector function.
- ☐ I can take the integral of a vector function.

Limits were developed to formalize the idea of a derivative and an integral. Now that we have defined how limits work for vector functions, we know how to define how derivatives and integrals work.

Because limits distribute through to the components of the vector, the same is true for derivatives and integrals. That is, for

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

we have that

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

and

$$\int \vec{r}(t)\ dt = \left\langle \int f(t)\ dt, \int g(t)\ dt, \int h(t)\ dt \right\rangle$$

Here are some differentiation rules. They can come in handy when calculations get complicated or when you would like to prove something. In the following rules, $\vec{u}(t)$ and $\vec{v}(t)$ are differentiable vector functions, $c$ is a scalar, and $f$ is a real-valued function (takes in a scalar and produces a scalar).

1. $\dfrac{d}{dt}\left[\vec{u}(t) + \vec{v}(t)\right] = \vec{u}\,'(t) + \vec{v}\,'(t)$
2. $\dfrac{d}{dt}\left[c\vec{u}(t)\right] = c\vec{u}\,'(t)$
3. $\dfrac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$ (like the product rule)
4. $\dfrac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$ (also like the product rule)
5. $\dfrac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}\,'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}\,'(t)$ (also like the product rule)

6. $\dfrac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\vec{u}\,'(f(t))$ (like the chain rule)

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### 4.2.1 Examples

**Example 4.2.1.1** *Find the derivative of the following function*

$$\vec{r}(t) = \left\langle \tan t, \frac{t}{1+t}, \cosh(t)\sinh(t) \right\rangle$$

We just take the derivatives of each component. Don't forget things like the quotient rule and the product rule.

$$\begin{aligned}\vec{r}\,'(t) &= \left\langle \sec^2 t, \frac{(1+t)-t}{(1+t)^2}, \cosh(t)\cosh(t) + \sinh(t)\sinh(t) \right\rangle \\ &= \left\langle \sec^2 t, \frac{1}{(1+t)^2}, \cosh^2(t) + \sinh^2(t) \right\rangle\end{aligned}$$

**Example 4.2.1.2** *Find the integral of the following function*

$$\vec{r}(t) = \left\langle \tan t, \frac{1}{1+t^2}, \cosh(t)\sinh(t) \right\rangle$$

$$\begin{aligned}\int \vec{r}(t)dt &= \left\langle \int \tan t\ dt, \int \frac{1}{1+t^2}\ dt, \int \cosh(t)\sinh(t)\ dt \right\rangle \\ &= \left\langle \int \frac{\sin t}{\cos t}\ dt, \arctan(t), \int u\ du \right\rangle \\ &= \left\langle \int \frac{-1}{u}\ du, \arctan(t), \frac{u^2}{2} \right\rangle \\ &= \left\langle -\ln|u|, \arctan(t), \frac{\cosh^2 t}{2} \right\rangle \\ &= \left\langle -\ln|\cos(t)|, \arctan(t), \frac{\cosh^2 t}{2} \right\rangle\end{aligned}$$

**Example 4.2.1.3** *Sketch the following curve and and its position vector at* $t = -1$.

$$\vec{r}(t) = \left\langle t - 2, t^2 + 1 \right\rangle$$

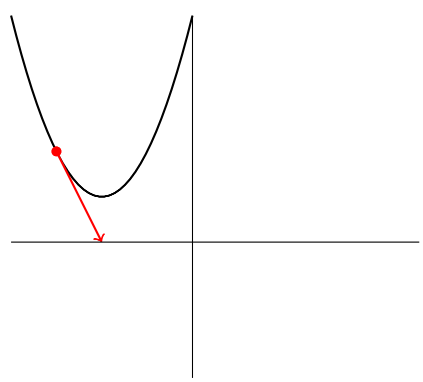
Let's first sketch the graph. Remember, to graph this curve, we'll write out a table for values of $t$ and find the corresponding pairs $(x(t), y(t))$. We then graph these pairs. That is,

| $t$ | $\vec{r}(t)$ |
|---|---|
| $-2$ | $\langle -4, 5 \rangle$ |
| $-1$ | $\langle -3, 2 \rangle$ |
| $0$ | $\langle -2, 1 \rangle$ |
| $1$ | $\langle -1, 2 \rangle$ |
| $2$ | $\langle 0, 5 \rangle$ |

To find the tangent vector, we find the derivative of $\vec{r}(t)$ at $t = -1$ and graph. Notice that

$$\vec{r}'(t) = \langle 1, 2t \rangle \implies \vec{r}'(-1) = \langle 1, -2 \rangle$$

Then, at the point $t = -1$, or $(-3, 2)$, we draw the vector $\langle 1, -2 \rangle$.



**Summary of Ideas: Derivatives and Integrals of Vector Functions**

- The **derivative** of a vector function is calculated by taking the derivatives of each component.

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

- The **integral** of a vector function is calculated by taking the integral of each component.

$$\int \vec{r}(t)\ dt = \left\langle \int f(t)\ dt, \int g(t)\ dt, \int h(t)\ dt \right\rangle$$

- There are differentiation rules similar to the product rule and the chain rule that apply to our new functions like the cross product and the dot product.