

3.6 Cylinders and Quadric Surfaces

Objectives

- I know the definition of a cylinder.
- I can name the 6 quadric surfaces, write their equation, and sketch their graph.

Let's take stock in the types of equations we're familiar with:

- A sphere centered at (a, b, c) with radius r

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

- A line through the point (p_1, p_2, p_3) in the direction $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$x(t) = p_1 + v_1 t$$

$$y(t) = p_2 + v_2 t$$

$$z(t) = p_3 + v_3 t$$

- A plane with normal vector $\vec{n} = \langle n_1, n_2, n_3 \rangle$ through the point (p_1, p_2, p_3)

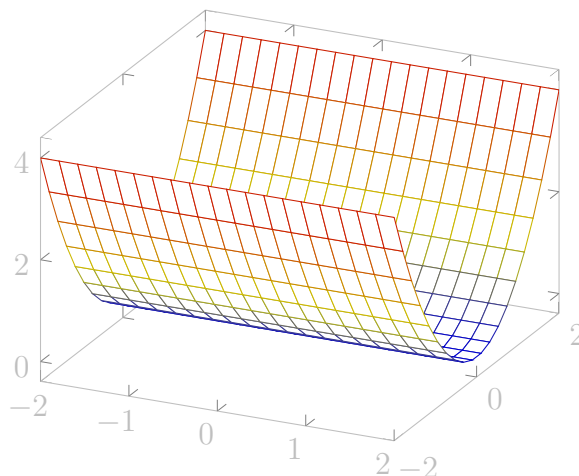
$$n_1(x - p_1) + n_2(y - p_2) + n_3(z - p_3) = 0$$

We all know, however, there are more interesting shapes that exist in three dimensions.

For example, **cylinders** are surfaces that are created from parallel lines (called **rulings**). We typically think of a soda can type shape (that goes up and down forever)



but there are other varieties like planes and a parabolic sheet (pictured below).



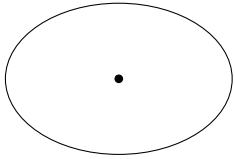
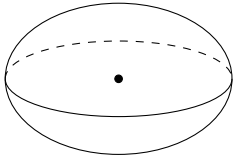
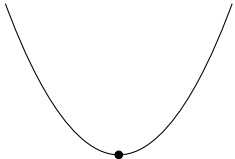
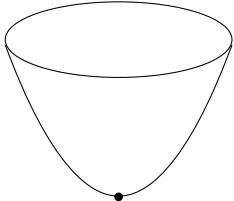
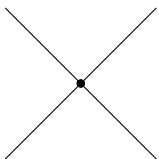
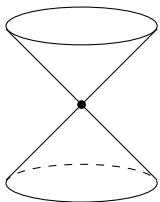
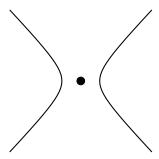
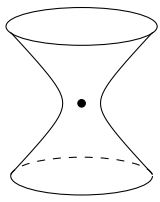
Notice that the picture above is made with two kinds of paths. One set are parabolas and the second set are straight lines. These straight lines are what makes this figure a **cylinder** (as defined in this course).

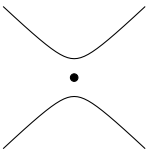
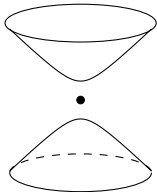
Another class of interesting surfaces are **quadric surfaces**, which we can think of as three-dimensional versions of conic surfaces. In fact, to help inform us about these equations, we will talk about them alongside conics.

You will not be responsible for knowing the conics, however, if you remember them, this will help you remember the three-dimensional equivalents. If you do not remember them, this is an excellent opportunity to remind yourself of these objects.

In the the examples below, the center point of the picture will always be the point (a, b) or (a, b, c) , depending on the dimension. The numbers $p, q,$ and r are positive constants.

2D Shape	2D Graph	3D Shape	3D Graph
Circle $(x - a)^2 + (y - b)^2 = r^2$		Sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$	

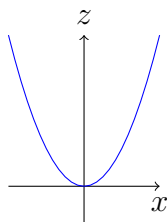
2D Shape	2D Graph	3D Shape	3D Graph
<p>Ellipse</p> $\frac{(x - a)^2}{p^2} + \frac{(y - b)^2}{q^2} = 1$		<p>Ellipsoid</p> $\frac{(x - a)^2}{p^2} + \frac{(y - b)^2}{q^2} + \frac{(z - c)^2}{r^2} = 1$	
<p>Parabola</p> $\frac{(x - a)^2}{p^2} = \frac{(y - b)}{q}$		<p>Elliptic Paraboloid</p> $\frac{(x - a)^2}{p^2} + \frac{(y - b)^2}{q^2} = \frac{(z - c)}{r}$	
<p>Double Line</p> $\frac{(x - a)^2}{p^2} = \frac{(y - b)^2}{q^2}$		<p>Cone</p> $\frac{(x - a)^2}{p^2} + \frac{(y - b)^2}{q^2} = \frac{(z - c)^2}{r^2}$	
<p>Hyperbola</p> $\frac{(x - a)^2}{p^2} - \frac{(y - b)^2}{q^2} = 1$		<p>Hyperboloid of one sheet</p> $\frac{(x - a)^2}{p^2} + \frac{(y - b)^2}{q^2} - \frac{(z - c)^2}{r^2} = 1$	

2D Shape	2D Graph	3D Shape	3D Graph
Hyperbola		Hyperboloid of two sheets	
$-\frac{(x-a)^2}{p^2} + \frac{(y-b)^2}{q^2} = 1$		$-\frac{(x-a)^2}{p^2} - \frac{(y-b)^2}{q^2} + \frac{(z-c)^2}{r^2} = 1$	

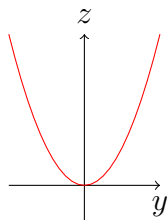
You may remember solids of revolution from calculus. That was where you rotated a curve about an axis to create a three-dimensional object. In some sense, we’re doing that when going from the 2D picture to the 3D picture. You can think of it as bubbling outward.

One way to understand these shapes is by looking at two-dimensional slices of the function, which are called **traces**.

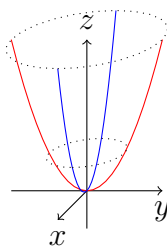
Consider the function $x^2 + y^2 = z$. If we set $y = 0$, we get the trace $x^2 = z$.



If we set $x = 0$, we get the trace $y^2 = z$.

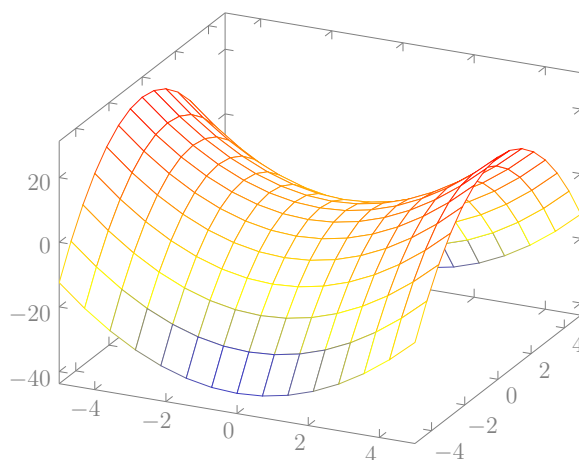


Finally, if $z = 0$, then we get the trace $x^2 + y^2 = 0$. But this is only true when $x = 0$ and $y = 0$. Therefore, we just get the point $(0, 0, 0)$. If we “glue” all these together, we get a paraboloid. That is



If you look at each function in our chart, they can be broken down into combinations of different conic traces.

There is one figure that is not in our chart, but is also a quadric. This is called the **hyperbolic paraboloid** and its created by two paraboloids of opposite signs and a hyperbola. It looks like the picture below.



It is represented by the equation

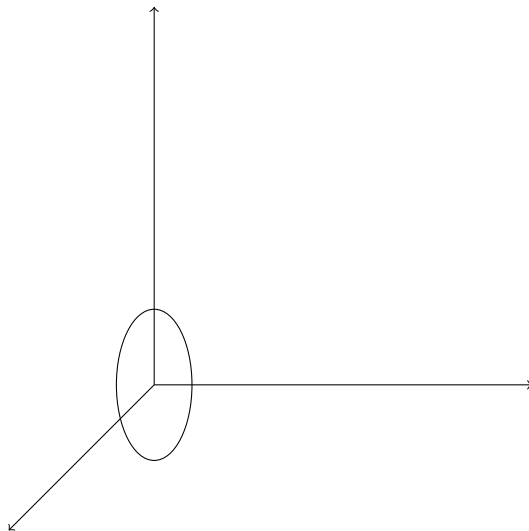
$$\frac{(x - a)^2}{p^2} - \frac{(y - b)^2}{q^2} = \frac{(z - c)}{r}$$

3.6.1 Examples

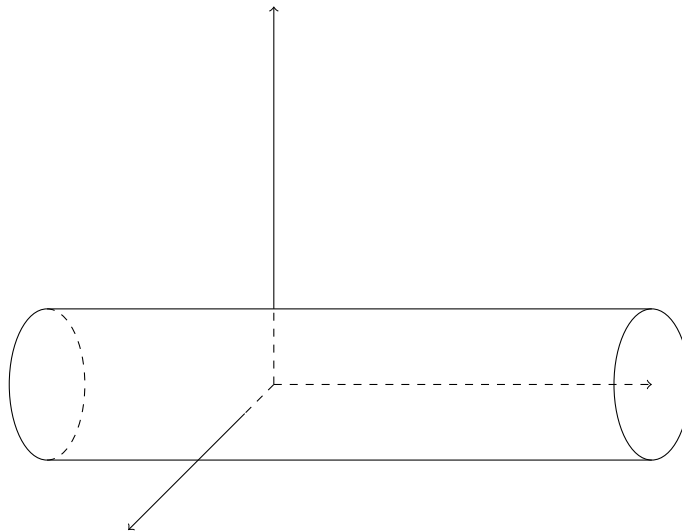
Example 3.6.1.1 *Describe and sketch the surface.*

$$x^2 + z^2 = 1$$

Whenever we are missing a variable in an equation, we know this will be a cylinder equation. Let's first graph the equation if $y = 0$.



Remember that this is just the circle and not the interior. Now, we consider if y could equal any value. Then the picture changes. Remember, dashed lines are lines you shouldn't be able to see, but are given as guides to give you a sense of depth.



It's important to notice that this is just the outer shell of the cylinder, not the solid cylinder. Can you see why? It is made based on the circle in the first picture. If we wanted to describe a solid cylinder, we would need an inequality. Specifically, it would be

$$x^2 + z^2 \leq 1$$

Example 3.6.1.2 Reduce the equation to one of the standard forms, classify the surface, and sketch it.

$$4y^2 + z^2 - x - 16y - 4z + 20 = 0$$

To solve this, we will have to complete the square. The first step is to organize the equation by variable and factor out coefficients of the highest degree term.

$$4(y^2 - 4y) + z^2 - 4z - x + 20 = 0$$

Now we add and subtract the necessary constants to complete the square. Then we simplify.

$$4(y^2 - 4y + 4 - 4) + z^2 - 4z + 4 - 4 - x + 20 = 0$$

$$4(y^2 - 4y + 4) - 16 + (z^2 - 4z + 4) - 4 - x + 20 = 0$$

$$4(y - 2)^2 - 16 + (z - 2)^2 - 4 - x + 20 = 0$$

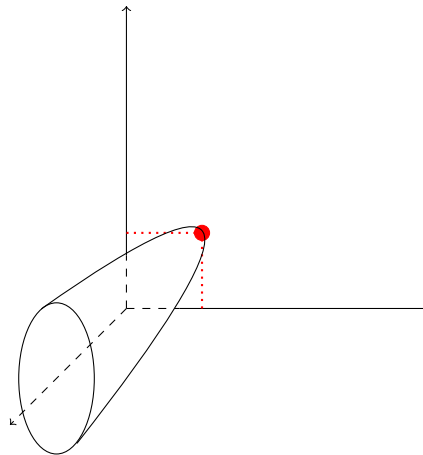
$$4(y - 2)^2 + (z - 2)^2 - x = 0$$

$$4(y - 2)^2 + (z - 2)^2 = x$$

Our final equation is

$$(y - 2)^2 + \frac{(z - 2)^2}{4} = \frac{x}{4}$$

This is an elliptic paraboloid centered at $(0, 2, 2)$.



Notice that the orientation is different than the chart. This is because our x direction acts as the z direction in the chart (compare the formulas to see why).

Summary of Ideas: Cylinders and Quadric Surfaces

- **Cylinders** are surfaces that are created from parallel lines (called **rulings**). They come in more shapes than the “soda can” shape we are most familiar with. They are created when one variable is allowed to be anything.
- **Quadrics** are 3-dimensional analogs of *conics*. They are
 - Spheres
 - Elliptic Paraboloids
 - Cones
 - Hyperboloids of one sheet
 - Hyperboloids of two sheets
 - Hyperbolic Paraboloid (or Saddle)
- To graph these, we must figure out the center and the orientation (it won't always be the z -axis, as we saw in an example).
- You will need to know how to complete the square to classify these curves.