5.6 The Chain Rule and Implicit Differentiation

Objectives

- ☐ I can how to use the chain rule to find the derivative of a function with respect to a parameter (like time).
- ☐ I know to write the result of the derivative in terms of the parameter(s) *only*.
- ☐ I can use implicit differentiation to determine the derivative of a variable with respect to another.

Up to this point, we have focused on derivatives based on space variables x and y. In practice, however, these spacial variables, or **independent variables**, are dependent on time. Therefore, it is useful to know how to calculate the function's derivative with respect to time. This requires the chain rule.

Let us remind ourselves of how the chain rule works with two dimensional functionals. If we are given the function $y = f(x)$, where x is a function of time: $x = g(t)$. Then the derivative of y with respect to t is the derivative of y with respect to x multiplied by the derivative of x with respect to t

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

The technique for higher dimensions works similarly. The only difficulty is that we need to consider *all* the variables dependent on the relevant parameter (time t).

1. **Chain Rule - Case 1**: Suppose $z = f(x, y)$ and $x = g(t), y = h(t)$. Based on the one variable case, we can see that dz/dt is calculated as

   $$\frac{dz}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt}$$

   In this context, it is more common to see the following notation.

   $$f_x = \frac{\partial f}{\partial x}$$

   The symbol ∂ is referred to as a "partial," short for partial derivative.

2. **Chain Rule - Case 2**: Parametric equations may have more than one variable, like t and s. This could be time and arc length, for example. In this case, $z = f(x, y)$ and $x = g(s, t), y = h(s, t)$. Then, every derivative is a partial derivative. Our formula for this situation is

$$z_t = f_x x_t + f_y y_t$$

What we do is take the derivative with respect to each variable, then take the derivative with respect to the hidden parametric variable. Written in our new notion, we have the equation

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Similarly if, we wanted to look at the derivative with respect to s, we'd have

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

To better understand these techniques, let's look at some examples.

14.5.1 Examples

Example 5.6.0.3 1. *Use the chain rule to find dz/dt for*

$$z = \cos(4x + y), \quad x = 5t^4, y = \frac{1}{t}$$

We begin by finding all the necessary derivatives. We put everything in terms of t by plugging in the functions for x and y. This gives us the following equations:

$$\frac{\partial z}{\partial x} = -4\sin(4x + y) = -4\sin(4[5t^4] + [1/t]) = -4\sin\left(\frac{20t^4 + 1}{t}\right)$$
$$\frac{\partial z}{\partial y} = -\sin(4x + y) = -\sin(4[5t^4] + [1/t]) = -\sin\left(\frac{20t^4 + 1}{t}\right)$$
$$\frac{dx}{dt} = 20t^3$$
$$\frac{dy}{dt} = -\frac{1}{t^2}$$

Now, we just plug into the formula

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

to get

$$\frac{dz}{dt} = \boxed{-80t^3 \sin\left(\frac{20t^4+1}{t}\right) + \frac{1}{t^2}\sin\left(\frac{20t^4+1}{t}\right)}$$

Example 5.6.0.4 2. *Use the chain rule to find $\partial z/\partial s$ for $z = x^2y^2$ where $x = s\cos t$ and $y = s\sin t$*

As we saw in the previous example, these problems can get tricky because we need to keep all the information organized. Let's walk through the solution of this exercise slowly so we don't make any mistakes. Our final answer will be in terms of s and t *only*.

First, we need to find the partials of z.

- $\dfrac{\partial z}{\partial x} = 2xy^2$
- $\dfrac{\partial z}{\partial y} = 2x^2y$

Next, we find the partials of the variables

- $\dfrac{\partial x}{\partial s} = \cos t$
- $\dfrac{\partial y}{\partial s} = \sin t$

Now, we plug in what we found into our equations.

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial s} = \left[2xy^2\right][\cos t] + \left[2x^2y\right][\sin t]$$

We also want to substitute and simplify so our answer is sensible. That gives us

$$\frac{\partial z}{\partial s} = \left[2(s\cos t)(s\sin t)^2\right][\cos t] + \left[2(s\cos t)^2(s\sin t)\right][\sin t] = 2s^3\cos^2 t\sin^2 t + 2s^3\cos^2 t\sin^2 t$$

We can use the trigonometric formula that $2\cos t\sin t = \sin 2t$ to get

$$\frac{\partial z}{\partial s} = \boxed{s^3\sin^2 2t}$$

Example 5.6.0.5 3. *Use the chain rule to find $\partial z/\partial t$ for $z = x^2y^2$ where $x = s\cos t$ and $y = s\sin t$ (the same equation for example 2).*

The math works similarly for $\frac{\partial z}{\partial t}$. We can use our work above for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. What's left are the partial derivatives of the "inner" equations, i.e.

- $\frac{\partial x}{\partial t} = -s\sin t$
- $\frac{\partial y}{\partial t} = s\cos t$

Now we just plug in as before

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\\ \frac{\partial z}{\partial t} &= \left[2xy^2\right][-s\sin t] + \left[2x^2y\right][s\cos t]\end{aligned}$$

If we substitute x and y for their parametric formulas, we get

$$\frac{\partial z}{\partial t} = \left[2(s\cos t)(s\sin t)^2\right][-s\sin t] + \left[2(s\cos t)^2(s\sin t)\right][s\cos t] = -2s^3\cos t\sin^3 t + 2s^3\cos^3 t\sin t$$

Again, using $2\cos t\sin t = \sin 2t$ to simplify as well as $\cos^2 t - \sin^2 t = \cos 2t$, we get the following result

$$\begin{aligned}\frac{\partial z}{\partial t} &= -2s^3\cos t\sin^3 t + 2s^3\cos^3 t\sin t\\ &= -s^3(2\sin t\cos t)\sin^2 t + s^3(2\sin t\cos t)\cos^2 t\\ &= s^3\sin 2t(\cos^2 t - \sin^2 t)\\ &= \boxed{s^3\sin 2t\cos 2t}\end{aligned}$$

Not all functions can be nicely written in "$z =$" form. In these situations, however, you may want to determine the partial derivative of z with respect to the variables x or y. To achieve this, we will use a technique is called **implicit differentiation**.

Let's first consider what happens in the two-dimensional case. Suppose we have the equation

$$F(x, y) = 0.$$

Remember that we can always set a function of two variables to zero by simply moving everything to one side. If we wanted to find the **total derivative** of the function, we would find

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Recall that $dx/dx = 1$. To find dy/dx, we solve and get the following formula:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

In three dimensions, it works similarly. We assume that the derivatives of the input variables with respect to each other is zero (like $dx/dy = 0$). We may use the same process above to get similar results. That is, we get the formulas

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Let's look at some examples.

14.5.2 Examples

Example 5.6.0.6 1. *Find* $\frac{dy}{dx}$ *for*

$$y\cos x = x^2 + y^2$$

Let's begin by writing the function in the correct form.

$$0 = x^2 + y^2 - y\cos x$$

Now we find F_x and F_y.

$$F_x = 2x + y\sin x$$
$$F_y = 2y - \cos x$$

Then,

$$\boxed{\frac{dy}{dx} = -\frac{2x + y\sin x}{2y - \cos x}}$$

Example 5.6.0.7 2. *Find* $\frac{\partial z}{\partial y}$ *for*

$$4 = x^2 - y^2 + z^2 - 2z$$

We begin by putting the equation in the correct form, find the relevant partials, and then plug in.

$$0 = x^2 - y^2 + z^2 - 2z - 4, \qquad F_y = 2y, \quad F_z = 2z - 2$$

$$\boxed{\frac{\partial z}{\partial x} = -\frac{2y}{2z-2} = -\frac{y}{z-1}}$$

Example 5.6.0.8 3. *Find* $\frac{\partial z}{\partial x}$ *for*

$$x^2 - y^2 + z^2 - 2z = 4$$

Let's use the information from example 2. All we need to find is

$$F_x = 2x$$

Then

$$\boxed{\frac{\partial z}{\partial x} = -\frac{2x}{2z-2} = -\frac{x}{z-1}}$$

Summary of Ideas: Chain Rule and Implicit Differentiation

- Sometimes x and y are functions of one or more parameters. We may find the derivative of a function with respect to that parameter using the **chain rule**.
- The formulas for calculating such derivatives are

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad \text{and} \qquad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

- To calculate a partial derivative of a variable with respect to another requires **implicit differentiation**

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$