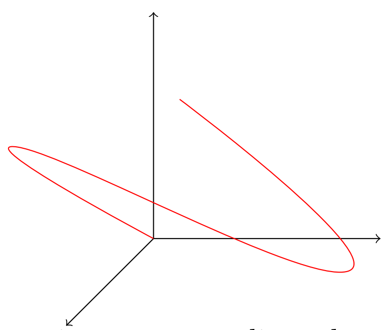
4.3 Arc Length and Curvature

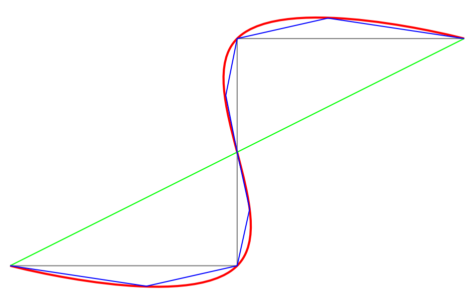
Objectives

- ☐ I know how to calculate arc length.
- ☐ I know how to find the unit tangent vector.
- ☐ I know two different three-dimensional equations for curvature and I know one two-dimensional equation for curvature.
- ☐ I know how to calculate the normal vector and the binormal vector.



A fundamental question in physics is, “How far did we go?” Since the paths travelled are described by displacement functions, which are vector functions, we want to come up with some kind of method of measuring this length. Looking at the picture left, how might you accomplish this?

Let’s begin with what we know. We know how to measure the distance of a straight line. If we approximate a complicated curved surface like the one in the picture with line segments, we could measure those line segments and approximate the distance. Consider the curve below. We can approximate it by a straight line connecting the end points (in green). That’s clearly an underestimate. We could also approximate the curve with three line segments (in gray), which is much better than the first one. Notice, that we can do better by using more line segments (like that in blue).



Let us recall the distance formula.

$$\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

We want to sum this over each line segment.

$$\sum_{i=0}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

If you recall the definition of an integral, then you'll notice that letting these line segments get infinitesimally small will produce an integral and derivatives! That is,

$$\text{Arc Length from } a \text{ to } b \ = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

Put another way,

$$\text{Arc Length from } a \text{ to } b \ = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\ dt$$

or

$$\text{Arc Length from } a \text{ to } b \ = \int_a^b |\vec{r}\,'(t)|\ dt$$

These equations aren't mathematically different. They are just different ways of writing the same thing.

4.3.1 Examples

Example 4.3.1.1 *Find the length of the curve*

$$\vec{r}(t) = \langle 3\cos(t), 3\sin(t), t\rangle$$

when $-5 \leq t \leq 5$.

However you choose to think about calculating arc length, you will get the formula

$$L = \int_{-5}^{5} \sqrt{(-3\sin(t))^2 + (3\cos(t))^2 + (1)^2} dt$$

We can simplify this integral with the equation

$$\sin^2 t + \cos^2 t = 1$$

That gives us

$$\begin{aligned}
L &= \int_{-5}^{5} \sqrt{9[\sin^2(t) + \cos^2(t)] + 1}\ dt \\
&= \int_{-5}^{5} \sqrt{9 + 1}\ dt \\
&= \int_{-5}^{5} \sqrt{10}\ dt \\
&= 5\sqrt{10} - (-5)\sqrt{10} = \boxed{10\sqrt{10}}
\end{aligned}$$

Example 4.3.1.2 *Find the length of the curve*

$$\vec{r}(t) = \left\langle \sqrt{2}t, e^t, e^-t \right\rangle$$

when $0 \leq t \leq 1$.

The formula tells us

$$L = \int_0^1 \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} dt$$

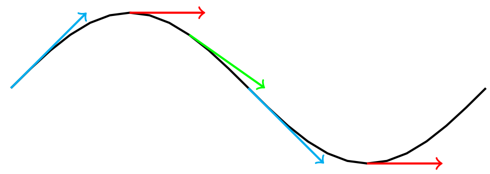
This integral looks very complicated. How can we solve it? We need to use some clever algebra.

$$\begin{aligned}
L &= \int_0^1 \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2}\ dt \\
&= \int_0^1 \sqrt{2 + e^{2t} + e^{-2t}}\ dt \\
&= \int_0^1 e^{-t}\sqrt{2e^{2t} + e^{4t} + 1}\ dt && \text{we factor out } e^{-2t}\text{, then take it out of the square root} \\
&= \int_0^1 e^{-t}\sqrt{(e^{2t} + 1)^2}\ dt \\
&= \int_0^1 e^{-t}(e^{2t} + 1)\ dt \\
&= \int_0^1 e^t + e^{-t}\ dt \\
&= \boxed{e + \frac{1}{e}}
\end{aligned}$$

In addition to length, we'd like to have some idea of the curvature of a path. For example, when probes are sent in outer space, engineers care a great deal about how many turns it must take since this impacts fuel consumption.

But how do we measure curvature? We want to measure the amount a curve changes direction. Since it is a measurement, it will be a scalar value.

When we imagine a curvy line like the one below, we see that the vector $\vec{r}\,'(t)$ in relation to the path created by $\vec{r}(t)$ changes direction wildly.

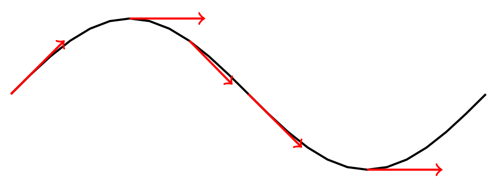


Notice, however, that not only do these vectors change direction, they also change in size! The blue vectors are longer than the green vector, which is longer than the red vectors. We don't want to measure changes in magnitude (which would correspond to speed), we only want to look at the direction.

Enter the **unit tangent vector**. This is a vector who is tangent to the curve but length 1. It is defined to be

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

When we graph the curve with the unit tangent vectors for the same values of t as above, we get a set of vectors that only change in direction.



We still have not addressed curvature. Curvature will be the amount $\vec{T}(t)$ changes as we travel along a segment. How can we measure this? If we think about it, this is the change of $\vec{T}(t)$ with respect to the changes in arc length *up to t*. This second function is represented as

$$s(t) = \int_a^t |\vec{r}\,'(u)|du$$

It is the same equation we had for arc length earlier except our end point is the variable t.

Therefore, **curvature**, $\kappa(t)$, is

$$\kappa(t) = \left|\frac{d\vec{T}}{ds}\right|$$

Remember, measurements are scalar values, so we need to take the *magnitude*.

Looking at this equation, we can make two observations:

1. It makes perfect sense as a definition for what is happening.

2. It's stupidly unusable.

Let's be honest, finding $s(t)$ is a pain and what does $d\vec{T}/ds$ really mean? We need to do a little more math to get a more usable equation.

First, remember that differentials work like fractions. That's one of the reasons they are written that way. Next, make the following observation:

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\frac{dt}{ds} = \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}$$

The fundamental theorem of calculus tells us

$$\text{if} \quad s(t) = \int_a^t |\vec{r}\,'(u)|du, \text{ then } \quad s'(t) = |\vec{r}\,'(t)|$$

Therefore, we get the following equation:

$$\frac{d\vec{T}}{ds} = \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} = \frac{\vec{T}\,'(t)}{|\vec{r}\,'(t)|}$$

Since curvature is the magnitude of the above vector, we get

$$\kappa(t) = \left|\frac{d\vec{T}}{ds}\right| = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$$

4.3.2 Examples

Example 4.3.2.1 *Let* $\vec{r}(t) = \langle t, 3\cos t, 3\sin t\rangle$. *Find the curvature of the path at time* t.

To calculate this, we're going to follow these steps

1. Find $\vec{r}\,'(t)$.
2. Find $|\vec{r}\,'(t)|$. Simplify when possible.
3. Find $\vec{T}(t)$.
4. Find $\vec{T}\,'(t)$.
5. Find $|\vec{T}\,'(t)|$. Simplify when possible.
6. Divide (5) by (2) to get $\kappa(t)$

We'll often get messy answers, but that's ok.

Let's do each step.

1. Find $\vec{r}\,'(t)$.

$$\vec{r}\,'(t) = \langle 1, -3\sin t, 3\cos t\rangle$$

2. Find $|\vec{r}\,'(t)|$. Simplify when possible.

$$\begin{aligned} |\vec{r}\,'(t)| &= \sqrt{1^2 + (-3\sin t)^2 + (3\cos t)^2} \\ &= \sqrt{1^2 + 9} \\ &= \sqrt{10} \end{aligned}$$

3. Find $\vec{T}(t)$.

$$\begin{aligned} \vec{T}(t) &= \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \\ &= \frac{\langle 1, -3\sin t, 3\cos t\rangle}{\sqrt{10}} \\ &= \left\langle \frac{1}{\sqrt{10}}, \frac{-3\sin t}{\sqrt{10}}, \frac{3\cos t}{\sqrt{10}} \right\rangle \end{aligned}$$

4. Find $\vec{T}\,'(t)$.

$$\vec{T}\,'(t) = \left\langle 0, \frac{-3\cos t}{\sqrt{10}}, \frac{3\sin t}{\sqrt{10}} \right\rangle$$

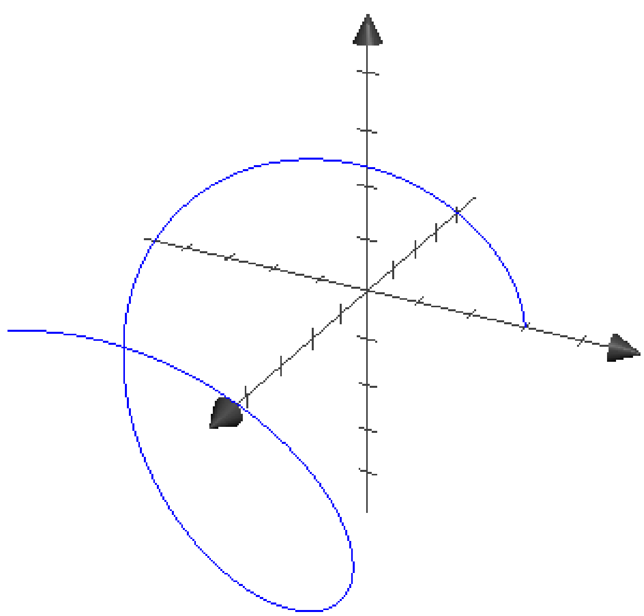
5. Find $|\vec{T}\,'(t)|$. Simplify when possible.

$$|\vec{T}\,'(t)| = \sqrt{0 + \frac{9\cos^2 t}{10} + \frac{9\sin^2 t}{10}} = \frac{3}{\sqrt{10}}$$

6. Divide (5) by (2) to get $\kappa(t)$

$$\kappa(t) = \frac{\frac{3}{\sqrt{10}}}{\sqrt{10}} = \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}} = \boxed{\frac{3}{10}}$$

So curvature for this equation is a nonzero constant. This means that at every time t, we're turning *in the same way* as we travel. The graph shows exactly this kind of movement



As you might guess, doing donuts with your car would also result in constant nonzero curvature.

Now, let's look at a messier example.

Example 4.3.2.2 *Let $\vec{r}(t) = \left\langle t, \frac{1}{2}t^2, t^2 \right\rangle$. Find the curvature of the path at time t.*

We will go through the steps. As you might expect, they will not simplify nicely.

1. Find $\vec{r}\,'(t)$.

$$\vec{r}\,'(t) = \langle 1, t, 2t \rangle$$

2. Find $|\vec{r}\,'(t)|$. Simplify when possible.

$$|\vec{r}\,'(t)| = \sqrt{1 + t^2 + 4t^2} = \sqrt{1 + 5t^2}$$

3. Find $\vec{T}(t)$.

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{1+5t^2}}, \frac{t}{\sqrt{1+5t^2}}, \frac{2t}{\sqrt{1+5t^2}} \right\rangle$$

4. Find $\vec{T}\,'(t)$.

   This is where is starts to get ugly. For the derivatives, I used product rule; however, you could use quotient rule as well. This might be a better approach since it keeps everything in one fraction. Also, keep in mind that $z(t)$ is just $2y(t)$. I will use this to do less work.

$$\vec{T}\,'(t) = \left\langle \frac{-10t}{2(\sqrt{1+5t^2})^3}, \frac{-10t^2}{2(\sqrt{1+5t^2})^3} + \frac{1}{\sqrt{1+5t^2}}, 2\left(\frac{-10t^2}{2(\sqrt{1+5t^2})^3} + \frac{1}{\sqrt{1+5t^2}} \right) \right\rangle$$

$$= \left\langle \frac{-5t}{(\sqrt{1+5t^2})^3}, \frac{-5t^2}{(\sqrt{1+5t^2})^3} + \frac{(\sqrt{1+5t^2})^2}{(\sqrt{1+5t^2})^3}, 2\left(\frac{-5t^2}{(\sqrt{1+5t^2})^3} + \frac{(\sqrt{1+5t^2})^2}{(\sqrt{1+5t^2})^3} \right) \right\rangle$$

$$= \left\langle \frac{-5t}{(\sqrt{1+5t^2})^3}, \frac{-5t^2+1+5t^2}{(\sqrt{1+5t^2})^3}, 2\left(\frac{-5t^2+1+5t^2}{(\sqrt{1+5t^2})^3} \right) \right\rangle$$

$$= \left\langle \frac{-5t}{(\sqrt{1+5t^2})^3}, \frac{1}{(\sqrt{1+5t^2})^3}, 2\left(\frac{1}{(\sqrt{1+5t^2})^3} \right) \right\rangle$$

$$= \frac{1}{(\sqrt{1+5t^2})^3} \langle -5t, 1, 2 \rangle$$

5. Find $|\vec{T}\,'(t)|$. Simplify when possible.

$$|\vec{T}\,'(t)| = \frac{1}{(\sqrt{1+5t^2})^3}|\langle -5t, 1, 2\rangle|$$

$$= \frac{1}{(\sqrt{1+5t^2})^3}\sqrt{25t^2+1+4}$$

$$= \frac{\sqrt{25t^2+5}}{(\sqrt{1+5t^2})^3}$$

$$= \frac{\sqrt{5}\sqrt{5t^2+1}}{(\sqrt{1+5t^2})^3}$$

$$= \frac{\sqrt{5}}{(\sqrt{1+5t^2})^2}$$

$$= \frac{\sqrt{5}}{1+5t^2}$$

Notice that I did one trick that made all my calculations easy. I factored out a constant and then took the magnitude of the simplified vector. If every entry has a constant in common, when we take the magnitude, the positive version of that constant comes out. Since our coefficient is always positive ($t^2 \geq 0$), we can do this without any problem.

If this is a confusion trick, don't worry. You will see this again in the review packet.

6. Divide (5) by (2) to get $\kappa(t)$

$$\kappa(t) = \frac{\frac{\sqrt{5}}{1+5t^2}}{\sqrt{1+5t^2}} = \frac{\sqrt{5}}{1+5t^2}\frac{1}{\sqrt{1+5t^2}} = \boxed{\frac{\sqrt{5}}{(\sqrt{1+5t^2})^3}}$$

Here is another formulation of curvature. Its derivation is less instructive, so I won't write it out.

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

Why use this formula? In some cases, it will be easier than the previous one. As you can see, finding $\vec{T}\,'(t)$ can be a pain! Also, we can use this formula for two-dimensional paths, like $y = f(x)$. In vector notation, we would represent $y = f(x)$ as

$$\vec{r}(t) = \langle x, f(x), 0\rangle$$

because $x = x$, $y = f(x)$, and z is not present so $z = 0$. If we plug in, we get

$$\vec{r}\,'(t) = \langle 1, f'(x), 0\rangle$$

and

$$\vec{r}\,''(t) = \langle 0, f''(x), 0\rangle$$

The cross product is then

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = |\langle 0, 0, f''(x)\rangle| = |f''(x)|$$

The denominator is

$$|\vec{r}\,'(t)|^3 = |\,\langle 1, f'(x), 0\rangle\,|^3 = |\sqrt{1 + (f'(x))^2}|^3 = |1 + (f'(x))^2|^{3/2}$$

So the curvature for a two-dimensional function $y = f(x)$ is

$$\kappa(t) = \frac{|f''(x)|}{|1 + (f'(x))^2|^{3/2}}$$

4.3.3 Examples

Example 4.3.3.1 *Find the curvature of* $\vec{r}(t) = t^3\vec{j} + t^2\vec{k}$.

We can, of course, use our previous formula; however, our cross-product formula will be easier. Why? First, we have a zero and zeros work well in cross products. Second, we have two different exponents that make finding $\vec{T}\,'(t)$ very difficult. Here are our steps:

1. Find $\vec{r}\,'(t)$.
2. Find $|\vec{r}\,'(t)|$. Simplify when possible.
3. Find $\vec{r}\,''(t)$.
4. Find $\vec{r}\,'(t) \times \vec{r}\,''(t)$.
5. Find $|\vec{r}\,'(t) \times \vec{r}\,''(t)|$. Simplify when possible.
6. Divide (5) by (2).

Let's go through them.

1. Find $\vec{r}\,'(t)$.

$$\vec{r}\,'(t) = 3t^2\vec{j} + 2t\vec{k} = \left\langle 0, 3t^2, 2t \right\rangle$$

2. Find $|\vec{r}\,'(t)|$. Simplify when possible.

$$|\vec{r}\,'(t)| = \sqrt{0 + 9t^4 + 4t^2} = \sqrt{t^2(9t^2+4)} = t\sqrt{9t^2+4}$$

3. Find $\vec{r}\,''(t)$.

$$\vec{r}\,''(t) = \langle 0, 6t, 2 \rangle$$

4. Find $\vec{r}\,'(t) \times \vec{r}\,''(t)$.

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3t^2 & 2t \\ 0 & 6t & 2 \end{vmatrix}$$

$$= (6t^2 - 12t^2)\vec{i} - (0)\vec{j} + (0)\vec{k}$$

$$= \left\langle -6t^2, 0, 0 \right\rangle$$

5. Find $|\vec{r}\,'(t) \times \vec{r}\,''(t)|$. Simplify when possible.

   Luckily, the math here is pretty easy. We know all constants should just come out as positive, so

$$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = 6t^2$$

6. Divide (5) by (2).

$$\kappa(t) = \frac{6t^2}{t\sqrt{9t^2+4}} = \boxed{\frac{6t}{\sqrt{9t^2+4}}}$$

Example 4.3.3.2 *Find the curvature of $y = x^2$.*

For this one, we will use the formula we derived. Namely,

$$\kappa(t) = \frac{|f''(x)|}{|1+(f'(x))^2|^{3/2}}$$

To do this, we'll find the first and second derivatives of $f(x)$ and plug in. That is,

$$f'(x) = 2x$$

and

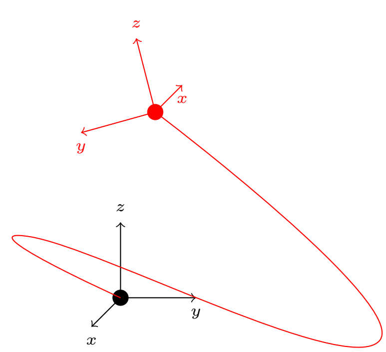
$$f''(x) = 2$$

so, we get

$$\kappa(t) = \frac{|2|}{|1+(2x)^2|^{3/2}} = \boxed{\frac{2}{(\sqrt{1+2x^2})^3}}$$

Given what we know about calculus and physics, we might believe that finding the unit tangent vector and the curvature is all that we need to do sophisticated physics. This is not so! In fact, we can go a few steps deeper into the math to develop the language for talking about different *frames of reference*. This is a mathematical way of describing different physical perspectives in space. For example, if you are on a roller coaster, your notions of up/down, left/right, and forward/backward change depending on where you are. At each location, you have a different physical viewpoint.

To develop this notion of a different physical viewpoint, we need to allow the origin to move and then find three new definitions of "moving in the x-direction," "moving in the y-direction," and "moving in the z-direction."



Imagine the path above is a segment of the roller coaster you are on. You begin at the black dot. Take note of the orientation of your sense of up/down(z), left/right (y), and forward/backward (x). You are pulled backwards, you then experience a quarter rotation to the left, you are pulled to your right, rotated a quarter to the left and pushed forward. You end at the black dot. Now your notion of the x, y, and z directions are complete different from when you started!

What is an easy way to construct these frames of reference as a particle travels? We can use the unit tangent vector! The unit tangent vector clearly indicates the forward(x) motion we experience. We only need two more unit vectors that describe the left/right (y) and up/down (z) directions.

The first is comes from $\vec{T}\,'(t)$. If a vector

$$\vec{v}(t)$$

has constant magnitude

$$|\vec{v}(t)|^2 = \vec{v}(t) \cdot \vec{v}(t) = c$$

then we could use the product rule for dot products (see Derivatives and Integrals of Functions) to get

$$2\vec{v}\,'(t) \cdot \vec{v}(t) = 0 \implies \quad \vec{v}\,'(t) \cdot \vec{v}(t) = 0$$

The same holds true for $\vec{T}(t)$. That is,

$$\vec{T}\,'(t) \cdot \vec{T}(t) = 0,$$

meaning they are orthogonal. While $\vec{T}(t)$ is a unit vector, $\vec{T}\,'(t)$ may not be. We want all the vectors to be normalized so we are only describing a direction. Therefore, our second vector is

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$$

This vector is called the **normal vector** because it is normal to our curve. We can think of it as describing left/right motion.

Now we have two vectors: one tangent to our path and one normal to it. We need three to have a new frame of reference. How can we get the third one? The cross product! This new vector is called the **binormal vector** since it is normal to *both* the tangent and normal vectors.

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The binormal vector can be thought of as the up/down direction.

Notice that $\vec{B}(t)$ is a unit vector. Why do we know this? Remember that

$$|\vec{B}(t)| = |\vec{T}(t)||\vec{N}(t)| \sin\theta = (1)(1)\sin\frac{\pi}{2} = 1$$

4.3.4 Examples

Example 4.3.4.1 *Find the tangent, normal, and binormal vector at the point* $(\sqrt{2}, \sqrt{2}, \pi/2)$ *for the equation*

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 2t \rangle$$

First, we should know for what value of t does $\langle 2\cos t, 2\sin t, 2t \rangle = \left\langle \sqrt{2}, \sqrt{2}, \pi/2 \right\rangle$. Setting the z components equal, we see that when $t = \pi/4$, we are at the desired point.

Next, we use our formulas and plug in our t value at the end. To find the tangent vector, $\vec{T}(t)$, we calculate

$$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 2 \rangle$$

and

$$\vec{r}\,'(t) = \sqrt{4\sin^2 t + 4\cos^2 t + 4} = 2\sqrt{2}$$

Then we divide.

$$\vec{T}(t) = \frac{\langle -2\sin t, 2\cos t, 2 \rangle}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle$$

Therefore, at the point $(\sqrt{2}, \sqrt{2}, \pi/2)$ (or $t = \pi/4$), the unit tangent vector is

$$\vec{T}(\pi/4) = \frac{1}{\sqrt{2}} \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1 \right\rangle = \boxed{\left\langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}} \right\rangle}$$

<u>Check:</u> Is this a unit vector? Yes!

$$|\vec{T}(\pi/4)| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{2}} = 1$$

Now let's find the unit normal vector. Remember that we need to use $\vec{T}(t)$ (not $\vec{T}(\pi/4)$) to do this.

$$\vec{T}\,'(t) = \frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle$$

The magnitude of that vector is the following:

$$|\vec{T}\,'(t)| = \frac{1}{\sqrt{2}} |\langle -\cos t, -\sin t, 0 \rangle| = \frac{1}{\sqrt{2}} \sqrt{\cos^2 t + \sin^2 t} = \frac{1}{\sqrt{2}}$$

Therefore,

$$\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$$

At the point $(\sqrt{2}, \sqrt{2}, \pi/2)$ (or $t = \pi/4$), the unit normal vector is

$$\boxed{\vec{N}(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle}$$

Check: Is this a unit vector? Yes!

$$|\vec{N}(\pi/4)| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

Finally, we can take the cross product of $\vec{T}(t) \times \vec{N}(t)$ and evaluate at $t = \pi/4$. Alternatively, we can take the cross product of $\vec{T}(\pi/4) \times \vec{N}(\pi/4)$. Either method will work.

$$\begin{aligned}
\vec{B}(\pi/4) &= \vec{T}(\pi/4) \times \vec{N}(\pi/4) \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1/2 & 1/2 & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 \end{vmatrix} \\
&= \boxed{\frac{1}{2}\vec{i} - \frac{1}{2}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}}
\end{aligned}$$

Summary of Ideas: Arc Length and Curvature

- We can measure the length of a segment from $t = a$ to $t = b$ with the formula

$$L = \int_a^b |\vec{r}\,'(t)|\;dt$$

- The **unit tangent vector**, $\vec{T}(t)$, is the unit vector tangent to the curve and defined to be

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

- Curvature, $\kappa(t)$, is the measure of how much a path curves. The two equivalent formulas are

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$$

and

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

The higher the value, the more it curves.

- The curvature for a two-dimensional function $y = f(x)$ is

$$\kappa(t) = \frac{|f''(x)|}{|1 + (f'(x))^2|^{3/2}}$$

The higher the value, the more it curves.

- The vector normal to any curve is called the **normal vector** and is

$$N(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$$

- The vector normal to both the unit tangent vector and the normal vector is called the **binormal vector**.

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$