# 3 Vectors and the Geometry of Space

Up until this point in your career, you've likely only done math in 2 dimensions. It's gotten you far in your problem solving abilities and you should be proud of all that you know.

Now, take a moment and look around you. What do you see? Think about all the objects around you. How many dimensions do they have? Almost every object you interact with has three dimensions, including the ground you walk on. We don't always walk on perfectly flat surfaces. Many times we walk up and down hills in addition to moving in the North/South direction and East/West direction. So it benefits us to know three-dimensional math. In fact, we also want to go beyond just three dimensions. We want to talk about moving in three-dimensional space while considering time.

In this chapter, we'll talk about navigating three-dimensional space mathematically. We will get to know the fundamental concepts necessary to understand calculus (chapter 14).

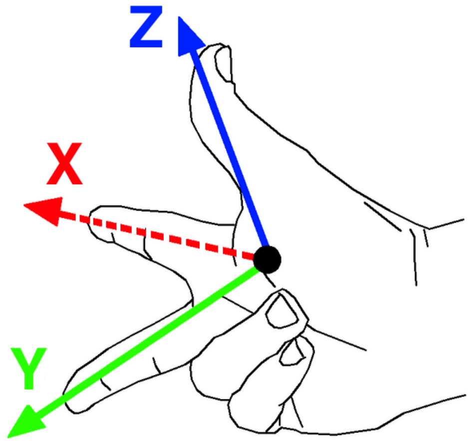
## 3.1 Three Dimensional Coordinate System

**Objectives**

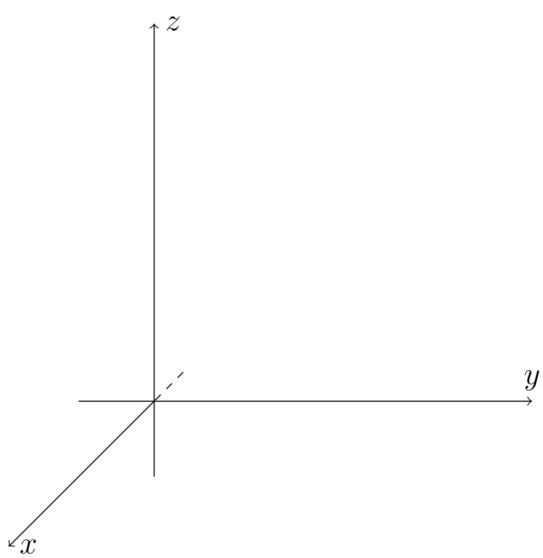
- ☐ I know how to use the right-hand rule.
- ☐ I know how to calculate distance in three dimensions.
- ☐ I know how to describe a sphere mathematically and I understand its connection to the distance formula.

The first step to communicating with another person is to speak the same language. In math, this means picking the same orientation. For this, we will use the **right-hand rule**. Here is how you use the rule:

1. Point your index finger of your right hand along the *positive* $x$-axis.
2. Orient your hand so that your middle finger points along the *positive* $y$-axis.
3. Now, extend your thumb. The direction it points is the *positive* $z$-axis.




Remember the dashed lines of the $x$-axis mean it is going slightly "into" the paper and the $y$-axis is going slightly out. The traditional way we will draw these axes will have the $x$-axis coming out of the paper instead, like the picture below.

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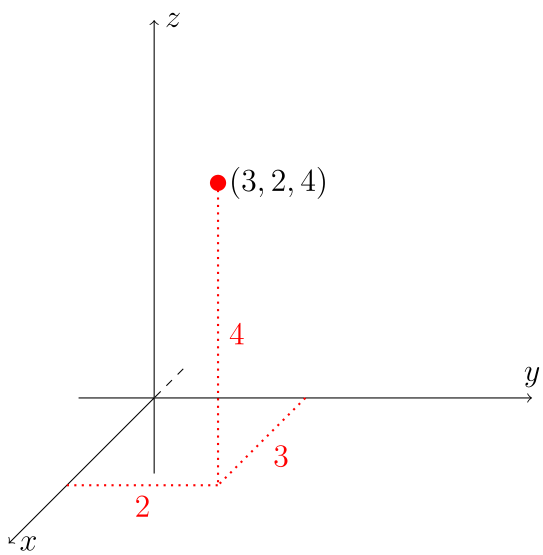
### 3.1.1 Examples

**Example 3.1.1.1** *Graph the point* $(3, 2, 4)$.

Graphing points works much like in two dimensions, except we also include depth. In the picture below, we move one of two ways.

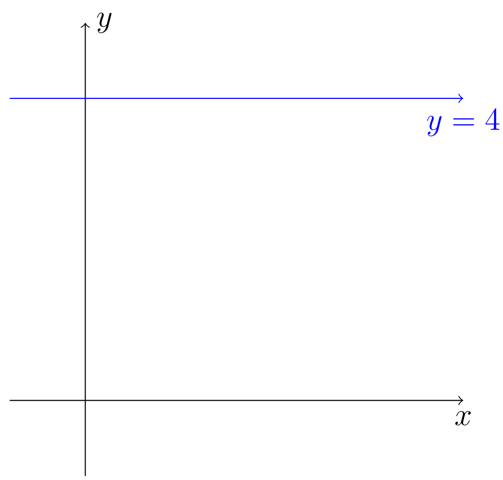
1. We move three steps along the $x$-axis, which brings us to $(3, 0, 0)$.
2. We, then, move two steps in the $y$-direction, which brings us to $(3, 2, 0)$.
3. Finally, we move 4 steps in the $z$-direction, which finishes our path to $(3, 2, 4)$.

Note that we can take this path in any order. We can move in the $y$ direction, then the $z$ direction, and finish with movement in the $x$-direction. You should do what you find easiest.

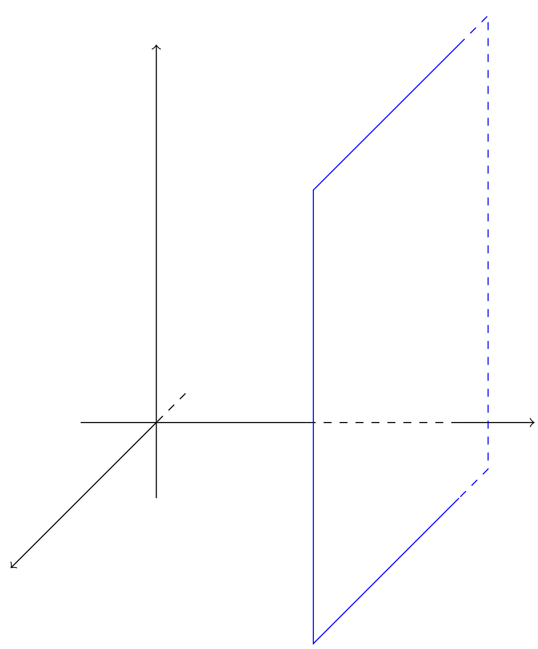



**Example 3.1.1.2** *Graph* $y = 4$.

Rather than simply graphing the function $y = 4$, let's motivate our understanding by considering its graph in two dimensions. That gives us a horizontal line. Remember, this line only fixes $y$, so $x$ can take on any value.




What will happen in 3 dimensions? In three dimensions, we have $x$, $y$, and $z$. If we fix a value for only $y$, then we are allowing $x$ and $z$ to be anything. What kind of object is that? It's called a **plane**.



Notice that in our picture, we draw edges for the plane. Is this correct? Does it, in fact, have edges? The answer is no. The plane goes on forever. We just can't draw that without giving our viewer some kind of idea what the object looks like. So we draw it as if it has edges, but know that it is line an infinitely long sheet of paper.

In the review section for these notes, we discussed distance in 2 dimensions. For points $A(x_1, y_1)$ and $B(x_2, y_2)$, the formula is

$$d(A,B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

In higher dimensions, the **distance formula** is the same. For points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the distance is

$$d(A,B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

The distance formula is used to describe the **sphere**, the 3-dimensional equivalent to the circle. Remember that a circle is the set of all points that are distance $r$ (the radius) away from a point $(a, b)$ (the center). That gives us

$$r = \pm\sqrt{(x-a)^2 + (y-b)^2} \implies r^2 = (x-a)^2 + (y-b)^2$$

The same holds for three dimensions. That is, for the **center** $(a, b, c)$ and **radius** $r$, the sphere is

$$r = \pm\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} \implies r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$$

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### 3.1.2 Examples

**Example 3.1.2.1** *Determine if the triangle formed by the points $A(3,-2,-3)$, $B(7,0,1)$, and $C(1,2,1)$ is equilateral, isosceles, or scalene. Is it a right triangle?*

To get any information about this triangle, we need to find the distances. Remember that an **equilateral triangle** is one with three equal sides, an **isosceles triangle** is one with two equal sides, and a **scalene triangle** is one where all sides are different.

The first step is to find the lengths of the sides.

$$\begin{aligned}
d(A,B) &= \sqrt{(3-7)^2 + (-2-0)^2 + (-3-1)^2} = \sqrt{16+4+6} = \sqrt{36} = 6 \\
d(A,C) &= \sqrt{(3-1)^2 + (-2-2)^2 + (-3-1)^2} = \sqrt{4+16+16} = \sqrt{36} = 6 \\
d(C,B) &= \sqrt{(1-7)^2 + (2-0)^2 + (1-1)^2} = \sqrt{36+4+0} = \sqrt{40} = 2\sqrt{10}
\end{aligned}$$

The triangle is clearly isosceles.

Is it a right triangle? We use the pythagorean theorem: $6^2 + 6^2 = 72 \neq 40$. So the answer is no.

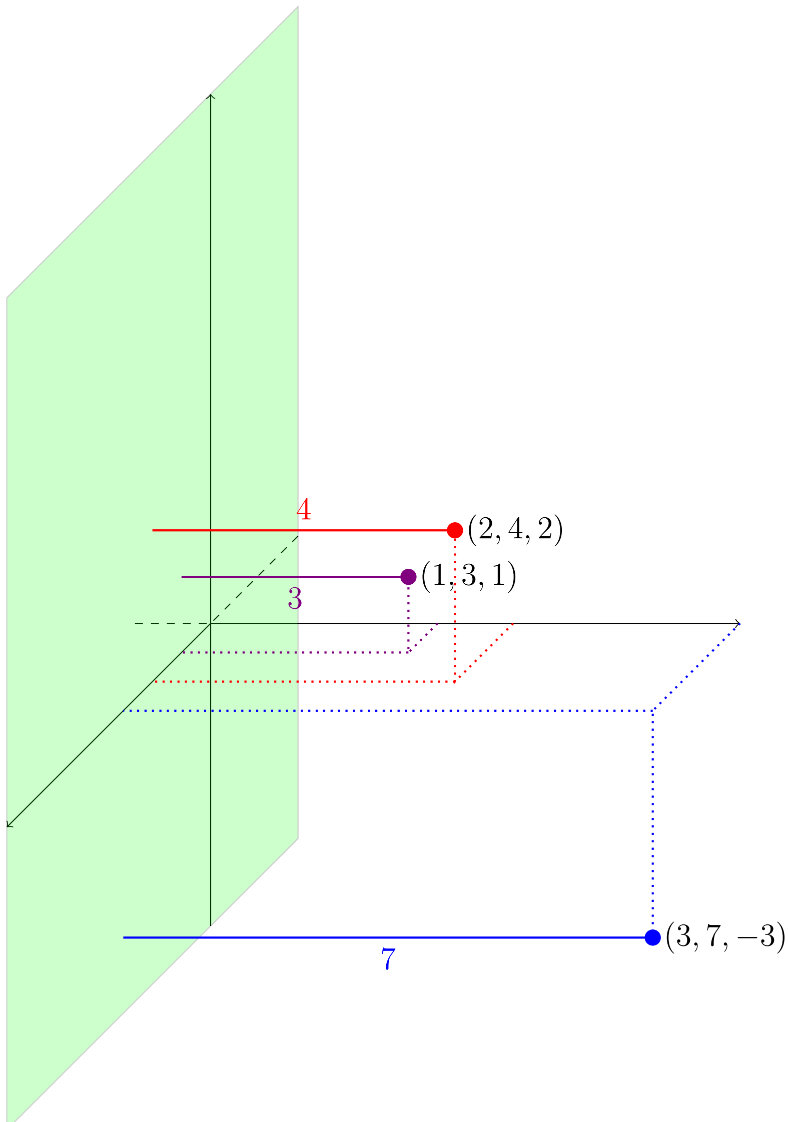
**Example 3.1.2.2** *Which of the following points are closest to the xz-plane?*

$$A(2,4,2),\ B(3,7,-3),\ C(1,3,1)$$

What equation can we use to describe the $xz$-plane? This plane consists of all the values of $x$ and $z$ while leaving $y$ fixed at 0. Therefore the equation we're looking at is $y = 0$. From here, we pick the smallest $y$ value, which is $\boxed{C(1,3,1)}$.

Why focus on $y$? The points $(2,0,2)$, $(3,0,-3)$, and $(1,0,1)$ are all in the plane. When we look at the distance formula, the smallest distance between the points and the plane will only consider the $y$ values.

To help us visualize the situation better, let's draw it.



**Example 3.1.2.3** *Define the sphere of radius* $4$ *centered at* $(1, 2, 3)$

Use the formula $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ to get

$$\boxed{16 = (x-1)^2 + (y-2)^2 + (z-3)^2}$$

**Example 3.1.2.4** *Determine the radius and center of the following sphere*

$$x^2 + y^2 + 2z^2 - 4x + 2y + 8z + 3 = 0$$

We first need to put this equation in the correct form. This requires that we complete the square. That is

$$\begin{aligned}
&0 = x^2 + y^2 + 2z^2 - 4x + 2y + 8z + 4 \\
\Longrightarrow\ &4 + 1 + 2(4) = (x^2 - 4x + 4) + (y^2 + 2y + 1) + 2(z^2 + 4z + 4) + 4 \\
\Longrightarrow\ &13 = (x-2)^2 + (y+1)^2 + 2(z+2)^2 + 4 \\
\Longrightarrow\ &9 = (x-2)^2 + (y+1)^2 + 2(z+2)^2
\end{aligned}$$

So the radius is $\boxed{3}$ and the center is $\boxed{(2, -1, -2)}$

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**Summary of Ideas: Three-dimensional Coordinate System**

- The **right-hand rule** is the orientation we use when we draw the $xyz$ coordinate axes.
- We plot points as we did in two dimensions, but we try to use slanted lines to depict depth.
- We define the distance between two points as

$$d(A, B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

- The formula for a sphere is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$