Geometry in Two Dimensions
Class Summary
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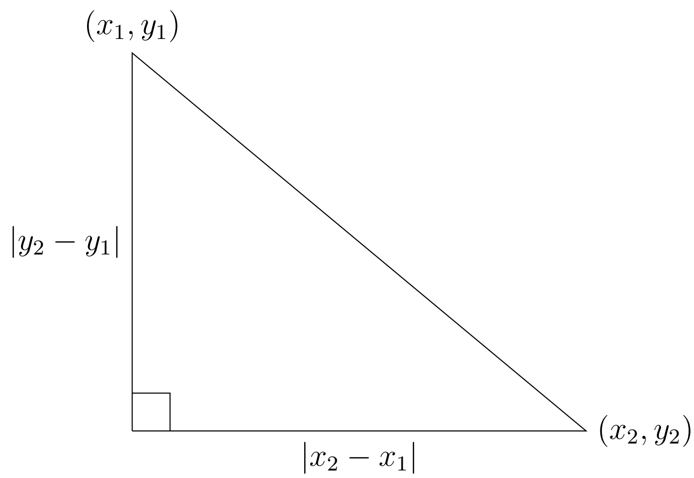
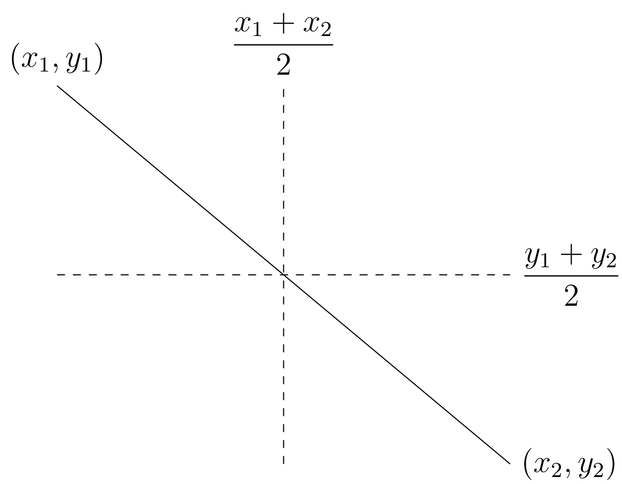
Let's start with the basics ...

in order to find the distance between points use the formula:
$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

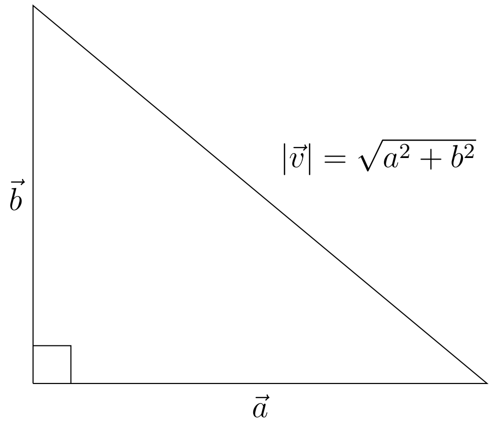


in order to find the midpoint between points in three dimensional space use the formula:
$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$



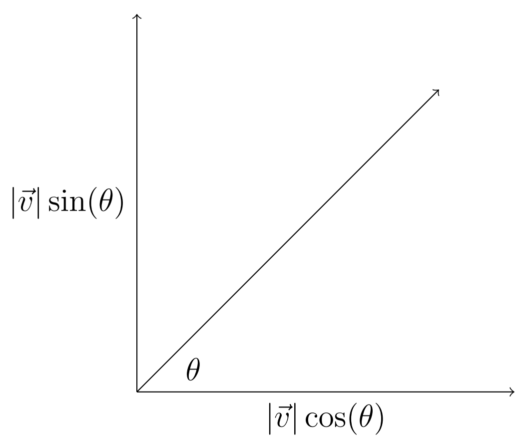
Now let's move onto some simple vectors ...

in order to find the magnitude of $\vec{v} = \langle a, b, c \rangle$ use the formula:
$|\vec{v}| = \sqrt{a^2 + b^2}$

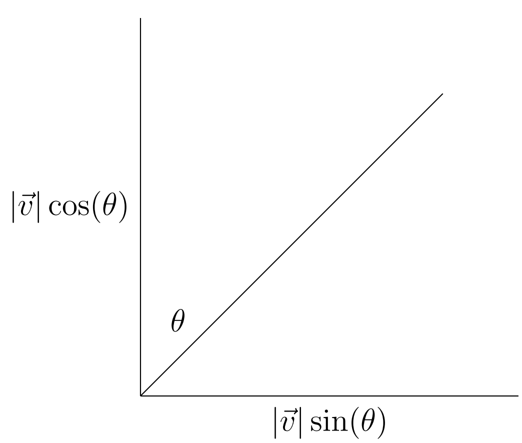


Note: if $|\vec{v}| = 1$ then $\vec{v}$ is called a *unit vector*

when you're given an angle measured from the horizontal axis . . .
horizontal component: $|\vec{v}|\cos(\theta)$
vertical component: $|\vec{v}|\sin(\theta)$



when you're given an angle measured from the vertical axis . . .
horizontal component: $|\vec{v}|\sin(\theta)$
vertical component: $|\vec{v}|\cos(\theta)$



Note: two points are considered *collinear* if their vectors are parallel

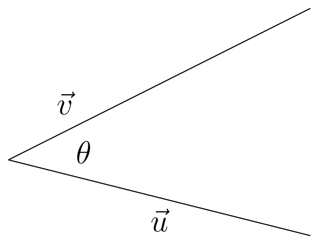
Finally we've arrived at the Dot Product . . .

the *Dot Product* is defined by:

$\cos(\theta) = \dfrac{\vec{v} \cdot \vec{u}}{|\vec{v}||\vec{u}|}$

but, when $\vec{v} = \langle a, b \rangle$ and $\vec{u} = \langle d, e \rangle$ it is commonly written as:

$\vec{v} \cdot \vec{u} = ad + be$



notice that a dot product will give a *scalar* answer, not a vector
Note: if $\vec{v} \cdot \vec{u} = 0$ then the vectors are *perpendicular* or *orthogonal*

the vector projection of $\vec{u}$ onto $\vec{v}$ is given by the formula:

$proj_{\vec{v}}\vec{u} = \left(\dfrac{\vec{v} \cdot \vec{u}}{|\vec{v}|^2}\right)\vec{v}$

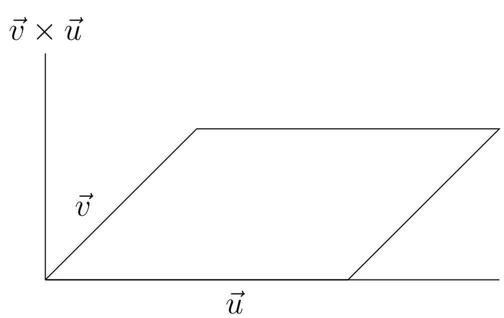
the scalar projection of $\vec{u}$ onto $\vec{v}$ is given by the formula:

$comp_{\vec{v}}\vec{u} = \left(\dfrac{\vec{v} \cdot \vec{u}}{|\vec{v}|}\right)$

But, even better than the Dot Product is the *Cross Product*. . .

The cross product of two vectors, $\vec{v} = \langle a, b, c \rangle \times \vec{u} = \langle d, e, f \rangle$ is defined as:

$$\vec{v} \times \vec{u} = |\vec{v}||\vec{u}| \sin(\theta) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ d & e & f \end{vmatrix} = (bf - ce)\vec{i} - (af - cd)\vec{j} + (ae - bd)\vec{k}$$



Notice that the Cross Product results in a **vector** *not a scalar.*
Also, note that the cross product is **not communative**.

Use the Cross Product to find the area of a parallelogram with sides made up of vectors $\vec{u}$ and $\vec{v}$:
$area = |\vec{u} \times \vec{v}|$

If you're trying to find a vector parallel to two other vectors, **cross them!**

Combine the Dot Product and the Cross Product to find the volume of a parallelepiped with sides made up of vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$...

$volume = |\vec{u} \cdot (\vec{v} \times \vec{w})|$

Let's move onto some lines ...

Lines in three dimensions are given by the parametric equations:

$x = x_0 + at$
$y = y_0 + bt$
$z = z_0 + ct$
where (x, y, z) is a point and $\langle a, b, c \rangle$ is a vector.

Lines in three dimensions are given by the symmetric equations:

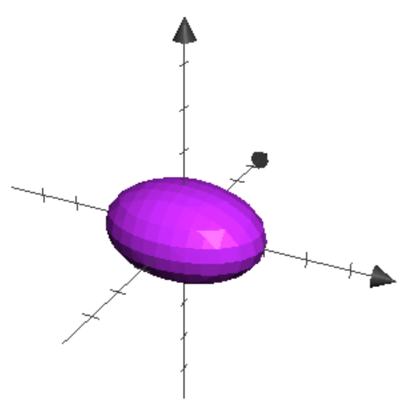
$t = \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$
where, again, (x, y, z) is a point and $\langle a, b, c \rangle$ is a vector.

The equation of a plane in three dimensions is given by the formula:

$a(x - x_0) + b(y - y_0) + c(z - z_0)$
where (x, y, z) is a point and $\langle a, b, c \rangle$ is a vector.

Now let's look at some 3D figures ...

Ellipsoid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



Elliptic Paraboloid: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z}{c}$

Hyperbolic Paraboloid: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \dfrac{z}{c}$

Cone: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$

Hyperboloid of One Sheet: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

Hyperboloid of Two Sheets: $\frac{-x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

More Calculus!

A ***Vector-Valued Function*** *is just a function written as a vector and is defined as*
$r(\vec{t}) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

Find the limit by
$\lim_{(x,y)\to(a,b)} f(x) = L$

Partial Derivative for x
$f_x(x,y) = \frac{\partial f}{\partial x}$
treat y as a constant, differentiate with respect to x
Partial Derivative for y
$f_y(x,y) = \frac{\partial f}{\partial y}$
treat x as a constant, differentiate with respect to y

The equation of the tangent plane is defined by
$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

If you want to find the ***Linear Approximation of*** **f** ***at point*** **(a, b)** *use the formula ...*
$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

The **directional derivative** *in the direction of unit vector $\vec{u} = \langle a, b \rangle$ is*
$D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$

The **gradient** *of f is the vector function ∇f given by*
$\nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$

Minimum and Maximum Values ...
First Derivative Test:

- If f has a local maximum, minimum, or saddle point at (x_0, y_0) then $\nabla f(x_0, y_0) = 0$
- (x_0, y_0) is called a **critical point**

Second Derivative Test:

- $D(x_0, y_0) = f_{xx} f_{yy} - (f_{xy})^2$
- if $D(x_0, y_0) > 0$ and $f_{xx} > 0$ then f has a local minimum

- if $D(x_0, y_0) > 0$ and $f_{xx} < 0$ then f has a local maximum
- if $D(x_0, y_0) < 0$ then (x_0, y_0) is a **saddle point**

Finally, we will end with **Lagrange Multipliers!**

Basically, we want the maximum and minimum values of $f(x, y, z)$ with constraint $g(x, y, z) = k$

- Find values of x, y, z and λ such that $\nabla f(x, y, z) = \lambda g(x, y, z)$
- Solve for x, y, and z in terms of λ
- Finally, solve for λ in the constraint