WHAT IS A WEDGE PRODUCT?

Before we can fully understand a wedge product, we must first know the symmetric group.

0.1 The Symmetric Group. The group S_n, or $\mathfrak{S}_n$ as it is denoted in class, is the group structure on functions which permute n elements.

Let us consider a set X of n distinct elements. Let $\mathcal{X}$ be the set of arrangements of elements in X, represented by n-tuples: $(1, \ldots, n)$ (without repeats). Elements of S_n are automorphisms on $\mathcal{X}$ that permute entries.

Exercise 1. Write out the automorphisms when $n = 3$.

Solution 1. There are 6 ways to permute 3 objects. We have arranged them in an order suitable to the image. The labelings of the automorphisms should be read left to right in a cyclic fashion. For example, (123) means $1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$.

- $id : (1, 2, 3) \mapsto (1, 2, 3)$
- $(23) : (1, 2, 3) \mapsto (1, 3, 2)$
- $(12) : (1, 2, 3) \mapsto (2, 1, 3)$
- $(123) : (1, 2, 3) \mapsto (2, 3, 1)$
- $(132) : (1, 2, 3) \mapsto (3, 1, 2)$
- $(13) : (1, 2, 3) \mapsto (3, 2, 1)$

Elements in the permutation group are either odd or even. The number arises based on on the number of transpositions needed to create the permutation. So if $s \in S_n$ requires m transpositions then $\text{sgn}(s) = (-1)^m$.

Exercise 2. Find the signs for each element in S_3.

Solution 2. There are 6 ways to permute 3 objects. We have arranged them in an order suitable to the image. The labelings of the automorphisms should be read left to right in a cyclic fashion. For example, (123) means $1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$.

- $\text{sgn}((id)) = (-1)^0 = 1$
- $\text{sgn}((23)) = (-1)^1 = -1$
- $\text{sgn}(12)) = (-1)^1 = -1$
- $\text{sgn}((123)) = \text{sgn}((23)(12)) = (-1)^2 = 1$
- $\text{sgn}((132)) = \text{sgn}((32)(13)) = (-1)^2 = -1$
- $\text{sgn}((13)) = (-1)^1 = -1$

0.2 Commutativity and Symmetry. Based on our last session, we know tensor products are not commutative maps. That is, in general $v \otimes w \neq w \otimes v$ and therefore the same is true for arbitrary tensors

$$\sum c_i v_i \otimes w_i \neq \sum c_i w_i \otimes v_i.$$

If we wanted to "make" the tensor product commutative, we'd need to impose an equivalence relation. For the sake of keeping the conversation simple, suppose we are working within the space $V \otimes V$ where $V = k^2$. Let us define $\mathcal{C}^2$ to be the subspace of $V \otimes V$ of

$$\text{span}\{v_1 \otimes v_2 - v_2 \otimes v_1\}.$$

When we quotient $V \otimes V$ by $\mathcal{C}^2$, we are imposing commutativity because $v_1 \otimes v_2 - v_2 \otimes v_1 = 0$ means $v_1 \otimes v_2 = v_2 \otimes v_1$.

When we consider tensor products that are higher than two, then $\mathcal{C}^k$ becomes bigger.

Exercise 3. What is the generating set of $\mathcal{C}^3$?

Solution 3. Remember that $\mathcal{C}^3$ must be the set of all vectors that generate the commutativity relation when equal to zero.

$$\begin{aligned}\mathcal{C}^3 = \text{span}\{&v_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_1 \otimes v_3,\\ &v_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_1 \otimes v_3,\\ &v_1 \otimes v_2 \otimes v_3 - v_3 \otimes v_2 \otimes v_1,\\ &v_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_1 \otimes v_3,\\ &v_1 \otimes v_2 \otimes v_3 - v_1 \otimes v_3 \otimes v_2\}\end{aligned}$$

Notice any relationship with S_3?

Definition 0.1. *The k^{th} symmetric power of V is the quotient space* $\text{Sym}^k(V) = (V^{\otimes k})/\mathcal{C}^k$.

Living inside this space are symmetric tensors.

Definition 0.2. *Let S_k be the symmetric group. A tensor $x \in V^{\otimes k}$ is* **symmetric** *if $s(x) = x$ for all $s \in S_k$.*

Theorem 0.3. $\text{Sym}^k(V) \cong \{x \in V^{\otimes k} \mid s(x) = x, \ \forall s \in S_k\}$.

Proof. Call $\mathcal{S}_k$ the symmetric subspace of $V^{\otimes k}$. Define the map $\rho_k : V^{\otimes k} \to \mathcal{S}_k$ such that $\rho_k\big|_{\mathcal{S}_k} = id_{\mathcal{S}_k}$. Then $\ker \rho_k = \mathcal{C}^k$, hence $\mathcal{S}_k \cong V^{\otimes k}/\mathcal{C}^k = \text{Sym}^k(V)$. □

Exercise 4. Let V be two dimensional and let $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$ be a basis for $V \otimes V$. Justify to yourself that $\{e_1 \otimes e_1, e_2 \otimes e_2, e_1 \otimes e_2 + e_2 \otimes e_1\}$ is a basis for $\text{Sym}^2(V)$. What is the dimension of $\text{Sym}^2(V)$?

Solution 4. Each basis element is invariant under the action of S_2, so they must generate a subspace (isomorphic to a subspace) of $\text{Sym}^2(V)$. The space

$$\text{Sym}^2(V) \cong \{x \in V \otimes V \mid s(x) = x, \ \forall s \in S_2\} \subset V \otimes V.$$

Since $\dim(V \otimes V) = 4$, $\dim(\text{Sym}^2(V)) \leq 2$. Since $\text{Sym}^2(V) \ncong V \otimes V$ (why?), $\sim (\text{Sym}^2(V)) \leq 3$. We have a subspace basis of three elements, so $\dim(\text{Sym}^2(V)) = 3$ and the basis we have is a basis for

$$\{x \in V \otimes V \mid s(x) = x, \ \forall s \in S_2\}.$$

Exercise 5. True of False:

$$\dim(\text{Sym}^k(V)) = \begin{pmatrix} n+k \\ k \end{pmatrix}$$

where $n = \dim V$. If true, sketch the proof. If false, give the correct dimension.

Solution 5. False.

$$\dim(\text{Sym}^k(V)) = \begin{pmatrix} n+k-1 \\ k \end{pmatrix}$$

Construct a basis for $\mathcal{S}_k = \{x \in V^{\otimes k} \mid s(x) = x, \ \forall s \in S_k\}$. From the possible n basis elements of V, we construct a basis for $\mathcal{S}_k$ by counting the number of ways we can pick k basis elements from the set of n of

V. For each choice, there is only one basis element of $\mathcal{S}_k$. For example, if $k = 3$ and $n = 2$, then our basis is:

$$\{e_1 \otimes e_1 \otimes e_1, e_2 \otimes e_2 \otimes e_2,$$
$$e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1,$$
$$e_2 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_2\}$$

Therefore, the number of basis elements is exactly the number of ways we can pick k elements from a set of n possible elements, allowing for repetitions.

Why care about $\mathrm{Sym}^k(V)$? There are a lot of "nice" facts about this space, but for this class, the reason to care is the following:

> If $\{x_1, \ldots, x_n\}$ is a basis for V, then any element of $\mathrm{Sym}^k(V)$ can be written as a polynomial of degree k in $x_1, \ldots, x_n$

In other words, $\mathrm{Sym}^k(V)$ is the space of polynomials of degree k in V. The new operation (i.e. the tensor product after the quotient) is equivalent to polynomial multiplication. We will denote this new operation with $\diamond$.

Exercise 6. Consider $(1, 1, 1) \diamond (-1, 1, -1) \diamond (0, 1, 0) \in \mathrm{Sym}^3(\mathbb{R}^3)$. What is the corresponding polynomial?

Solution 6. Let $x = (1, 0, 0)$, $y = (0, 1, 0)$, and $z = (0, 0, 1)$. Then

$$(x + y + z)(-x + y - z)(y) = -x^2y - 2xyz + y^3 - yz^2$$

is the corresponding polynomial.

Exercise 7. Let T be a linear operator on V (still two dimensional) such that $T(e_1) = e_1 + e_2$ and $T(e_2) = e_2$. Prove that the subspace of symmetric tensors $\mathrm{Sym}^2(V)$ is *invariant* under the *induced action* of T on $V \otimes V$.

Solution 7. Let us first describe the induced action of T on $V \otimes V$:

- $T(e_1 \otimes e_1) = (e_1 + e_2) \otimes (e_1 + e_2)$
- $T(e_1 \otimes e_2) = (e_1 + e_2) \otimes e_2$
- $T(e_2 \otimes e_1) = e_2 \otimes (e_1 + e_2)$
- $T(e_2 \otimes e_2) = e_2 \otimes e_2$

Now we wish to show that $\mathrm{Sym}^2(V)$ is invariant under the induced action. Consider $\mathcal{S}_2 \cong \mathrm{Sym}^2(V)$.

- $T(e_1 \otimes e_1) = e_1 \otimes e_1 + e_2 \otimes e_1 + e_2 \otimes e_1 + e_2 \otimes e_2$
- $T(e_1 \otimes e_2 + e_2 \otimes e_1) = e_1 \otimes e_2 + e_2 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2$
- $T(e_2 \otimes e_2) = e_2 \otimes e_2$

Notice that basis elements are simply mapped to combinations of basis elements. Hence, the space is invariant under the action.

Exercise 8. What is the induced action of T on polynomials $\mathrm{Sym}^2(\mathbb{R}^2)$?

Solution 8. $\mathrm{Sym}^2(\mathbb{R}^2)$ is also the space of polynomials of degree 2 in over $\mathbb{R}^2$. Hence, $e_1 \otimes e_1 \mapsto x^2$, $e_1 \otimes e_2 + e_2 \otimes e_1 \mapsto xy$, and $e_2 \otimes e_2 \mapsto y^2$. The induced action, is then

- $T(x^2) = x^2 + xy + y^2$
- $T(xy) = xy + 2y^2$

- $T(y^2) = y^2$

Exercise 9. Does there exist a linear operator $T : V \to V$ such that $\text{Sym}^2(V)$ is not invariant under the induced action?

Solution 9. No. The previous two exercises illustrate why.

Definition 0.4. $\text{Sym}(V) = \text{Sym}^0(V) \oplus \text{Sym}^1(V) \oplus \text{Sym}^2(V) \oplus \text{Sym}^3(V) \oplus \dots$ *is called the* **Symmetric Algebra**.

Recall that this is a graded algebra that corresponds to polynomials (this was mentioned two weeks ago). In that discussion, we used the fact that $\text{Sym}(V)$ is equivalent to the space of polynomials. Each $\text{Sym}^k(V)$ is a space of homogeneous polynomials of degree k.

Remark 0.5. Fun fact: The symmetric algebra $S(V)$ is the universal enveloping algebra of an abelian Lie algebra where the Lie bracket is 0. Now, go impress Nigel and Qijun!

0.3 Skew Symmetry and Wedge Product. A **symmetric** tensor $x \in V^{\otimes k}$ is such that $s(x) = x$ for all $s \in S_k$. A **skew symmetric** tensor is such that $s(x) = \text{sgn}(s)x$ for all $s \in S_k$.

Exercise 10. Let V be two dimensional. Show that

$$e_1 \otimes e_2 - e_2 \otimes e_1$$

is skew-symmetric.

Solution 10.

$$(12)[e_1 \otimes e_2 - e_2 \otimes e_1] = e_2 \otimes e_1 - e_1 \otimes e_2 = -e_1 \otimes e_2 + e_2 \otimes e_1 = -(e_1 \otimes e_2 - e_2 \otimes e_1)$$

Let $\mathcal{A}^k$ be the subspace of $V^{\otimes k}$

$$\text{span}\{v_1 \otimes \dots \otimes v_i \otimes \dots \otimes v_j \otimes \dots \otimes v_k \mid v_i = v_j \text{ for some } i \neq j\}.$$

Exercise 11. Consider a symmetric tensor in $x \in V^{\otimes k}$. Can x be written as a linear combination of elements in $\mathcal{A}^k$?

Solution 11. Yes. Each vector in $\mathcal{A}^k$ is invariant under a particular 2-cycle in S_k. Every element in S_k can be decomposed into a product of 2-cycles. Hence, every vector invariant under the action S_k lives in $\mathcal{A}^k$ (although it is much larger).

Definition 0.6. *The* k^{th} **exterior power** *of* V *is the quotient space* $\wedge^k(V) = (V^{\otimes k})/\mathcal{A}^k$.

Theorem 0.7. $\wedge^k(V) \cong \{x \in V \otimes V \mid s(x) = \text{sgn}(s)x \ \forall s \in S_k\}$

Proof. Call $\mathcal{R}_k$ the anti-symmetric subspace of $V^{\otimes k}$. Define the map $\varphi_k : V^{\otimes k} \to \mathcal{R}_k$ such that $\varphi_k\big|_{\mathcal{R}_k} = id_{\mathcal{R}_k}$. Then $\ker \varphi_k = \mathcal{A}^k$, hence $\mathcal{R}_k \cong V^{\otimes k}/\mathcal{A}^k = \wedge^k(V)$. ☐

Corollary 0.8. $\wedge^2(V) \cong \text{span}\{e_1 \otimes e_2 - e_2 \otimes e_1\}$

Proof. Since $\dim(V \otimes V) = 4$ and $\dim(\text{Sym}^3(V)) = 3$, then $\dim(\wedge^2 V) \leq 1$. Since $e_1 \otimes e_2 - e_2 \otimes e_1$ is skew-symmetric, $\dim(\wedge^2 V) \leq 1$. Hence $\dim(\wedge^2 V) = 1$. ☐

The tensor product under this quotient is often written as $\wedge$ (think of $\diamond$ used in the previous section) and called the **wedge product**. In particular,

$$v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_1.$$

This $\wedge$ is not commutative like $\diamond$, but it is **anti-commutative**, meaning

$$v_1 \wedge v_2 = -(v_2 \wedge v_1).$$

Proof. Notice that $x \wedge x = 0$ (why?). Therefore, $0 = (v_1+v_2)\wedge(v_1+v_2) = v_1\wedge v_1 + v_1\wedge v_2 + v_2 \wedge v_1 + v_2 \wedge v_2$. Hence $0 = v_1 \wedge v_2 + v_2 \wedge v_1$. □

Exercise 12. Prove this for arbitrary k.

Solution 12. We wish to show

$$v_1 \wedge \ldots \wedge v_i \wedge \ldots \wedge v_j \wedge \ldots \wedge v_k = -v_1 \wedge \ldots \wedge v_j \wedge \ldots \wedge v_i \wedge \ldots \wedge v_k.$$

Consider

$$\begin{aligned} 0 =& v_1 \wedge \ldots \wedge (v_i + v_j) \wedge \ldots \wedge (v_i + v_j) \wedge \ldots \wedge v_k \\ =& v_1 \wedge \ldots \wedge v_i \wedge \ldots \wedge v_i \wedge \ldots \wedge v_k + v_1 \wedge \ldots \wedge v_i \wedge \ldots \wedge v_j \wedge \ldots \wedge v_k + \\ & v_1 \wedge \ldots \wedge v_j \wedge \ldots \wedge v_i \wedge \ldots \wedge v_k + v_1 \wedge \ldots \wedge v_j \wedge \ldots \wedge v_j \wedge \ldots \wedge v_k \\ =& v_1 \wedge \ldots \wedge v_i \wedge \ldots \wedge v_j \wedge \ldots \wedge v_k + v_1 \wedge \ldots \wedge v_j \wedge \ldots \wedge v_i \wedge \ldots \wedge v_k \end{aligned}$$

which implies the result.

Exercise 13. True or False:

$$\dim(\wedge^k(V)) = \begin{pmatrix} n \\ k \end{pmatrix}$$

where $n = \dim V$.

Solution 13. True. One need only consider the basis elements of the wedge product. For $k \leq n$, basis elements are of the form $e_{s(1)} \wedge \ldots \wedge e_{s(k)}$ for all $s \in S_n$. This amounts to the number of unique ways I can pick k elements from a set of n.

Exercise 14. Let T be a linear operator on V (two dimensional) such that $T(e_1) = e_1 + e_2$ and $T(e_2) = e_2$. Is it true that $\wedge^2(V)$ is *invariant* under the *induced action* of T on $V \otimes V$?

Solution 14. The basis for $\wedge^2 V$ is $e_1 \wedge e_2$. The induced action is

$$T(e_1) \wedge T(e_2) = (e_1 + e_2) \wedge e_2 = e_1 \wedge e_2.$$

Therefore, the induced action is the identity. Hence, $\wedge^2 V$ is invariant under the induced action.

The wedge product is related to the determinant.

Exercise 15. Let $V = \mathbb{R}^2$, $e_1 = (1, 0)$, $e_2 = (0, 1)$. Reduce $(a, b) \wedge (c, d)$.

Solution 15. The problem becomes

$$\begin{aligned}(ae_1 + be_2) \wedge (ce_1 + de_2) &= ac(e_1 \wedge e_1) + bc(e_2 \wedge e_1) + ad(e_1 \wedge e_2) + bd(e_2 \wedge e_2)\\ &= -bc(e_1 \wedge e_2) + ad(e_1 \wedge e_2)\\ &= (ad - bc)(e_1 \wedge e_2)\end{aligned}$$

Hence, the wedge product of the row vectors of a 2×2 matrix yields the determinant. In fact, this holds for higher dimensions.

Exercise 16. Let $V = \mathbb{R}^2$, $e_1 = (1, 0), e_2 = (0, 1)$. Show that $(a, b) \wedge (c, d) \wedge (e, f) = 0$. Why does this answer make sense when thinking of volumes?

Solution 16. The problem reduces to

$$(ad - bc)(e_1 \wedge e_2) \wedge (ee_1 + fe_2) = e(ad - bc)(e_1 \wedge e_2 \wedge e_1) + f(ad - bc)(e_1 \wedge e_2 \wedge e_2) = 0$$

since $e_1 \wedge e_2 \wedge e_1 = -(e_1 \wedge e_1) \wedge e_2 = 0$. This makes sense since a 2×3 matrix has a two-dimensional image in a three-dimensional space and, therefore, has zero volume.

Exercise 17. Let $V = \mathbb{R}^3$, $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$. What is $(a, b, c) \wedge (d, e, f)$?

Solution 17. The problem becomes

$$\begin{aligned}(ae_1 + be_2 + ce_3) \wedge (de_1 + ee_2 + fe_3) &= ad(e_1 \wedge e_1) + bd(e_2 \wedge e_1) + cd(e_3 \wedge e_1)\\ &+ ae(e_1 \wedge e_2) + be(e_2 \wedge e_2) + ce(e_3 \wedge e_2)\\ &+ af(e_1 \wedge e_3) + bf(e_2 \wedge e_3) + cf(e_3 \wedge e_3)\\ &= (ae - bd)(e_1 \wedge e_2) + (af - cd)(e_1 \wedge e_3) + (bf - ce)(e_2 \wedge e_3)\end{aligned}$$

Notice that this is precisely the cross product of the vectors where $\vec{k} = (e_1 \wedge e_2), \vec{j} = (e_1 \wedge e_3), \vec{i} = (e_2 \wedge e_3)$.

Exercise 18. Let $V = \mathbb{R}^3$, $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$. What is $(a, b, c) \wedge (d, e, f) \wedge (g, h, i)$?

Solution 18. Using the previous problem, we get

$$[i(ad - bc) - h(af - cd) + g(ac - bd)](e_1 \wedge e_2 \wedge e_3),$$

which is precisely the determinant.

Definition 0.9. $\wedge(V) = \wedge^0(V) \oplus \wedge^1(V) \oplus \wedge^2(V) \oplus \wedge^3(V) \oplus \ldots$ *is called the* **Exterior Algebra**.

0.4 Subspaces of $V^{\otimes k}$.

So far, we have defined two subspaces of $V^{\otimes k}$:

$$\text{Sym}^k(V) \cong \{x \in V^{\otimes k} \mid s(x) = x, \forall s \in S_k\}$$

$$\wedge^k(V) \cong \{x \in V^{\otimes k} \mid s(x) = \text{sgn}(s)x, \forall s \in S_k\}$$

When $k = 2$, we saw in class that

$$V \otimes V \cong \text{Sym}^2(V) \oplus \wedge^2(V).$$

Exercise 19. Prove this using a basis argument.

Solution 19. For this proof, assume we are using the spaces isomorphic to $\text{Sym}^2(V)$ and $\wedge^2 V$, which live in $V \otimes V$. In previous exercises, we have show that $\dim \text{Sym}^2(V) = 3$ and $\dim \wedge^2 V = 1$. Since $\text{Sym}^2(V) \cap \wedge^2 V = \{0\}$, their direct sum is dimension 4. More over,

$$\text{Sym}^2(V) \oplus \wedge^2 V = \text{span}\{e_1 \otimes e_1, e_2 \otimes e_2, e_1 \otimes e_2 + e_2 \otimes e_1, e_1 \otimes e_2 - e_2 \otimes e_1\} = \text{span}\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}.$$

When $k = 3$, however, it is no longer true. That is,

$$V \otimes V \otimes V \ncong \text{Sym}^3(V) \oplus \wedge^3(V).$$

Exercise 20. Let $V = \mathbb{R}^3$. Find an element $x \in V \otimes V \otimes V$ such that $x \neq u \oplus v$ such that $s(u) = u$ and $s(v) = \text{sgn}(s)v$. (Hint: Use a basis argument.)

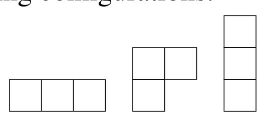
At the heart of why $k = 2$ is a direct sum of only two items while $k = 3$ is combinatorial. Hence the use of the **Young Tableau**.

A great introduction to Young tableaux can be found here: `http://www.ams.org/notices/200702/whatis-yong.pdf`. But we will quickly discuss how these objects work here.

Young tableaux begin with different ways in which one can arrange n boxes such that

- every column is longer than or equal to all columns to its right, and
- every row is longer than or equal to all rows below it.

If $n = 3$, then there are only the following configurations:



Inside these boxes are placed numbers such that

- entries (weakly) increase from left to right, and
- entries strictly increase from top to bottom.

To be continued...