Homework 48 Solutions :

Section 8.5 Trigonometric Equations

Note: A calculator is helpful on some exercises. Bring one to class for this lecture.

Some trigonometric equations are true for *all* values of the variable for which each trigonometric function is defined. We call these **identities.** We have verified many identities throughout this chapter. Other trigonometric equations are only true for *specific* values of the variable (or no values at all.) These are called **conditional trigonometric equations** or simply trigonometric equations.

Because of the periodic nature of trigonometric functions, we will often find that trigonometric equations have infinitely many solutions. We obviously cannot make a list of the infinitely many solutions. Therefore, we will describe these solutions using a formula (or formulas.) The most concise formula(s) describing the infinite set of solutions to a trigonometric equation is called the **general solution(s).** We will also be interested in finding the **specific solution(s)** on a given restricted interval. We will typically try to find the solutions on the interval $[0, 2\pi)$.

Read the online text and watch the animations to see a visual (graphical) representation of solutions to the trigonometric equation $\sin\theta = b$ for $-1 \le b \le 1$.

OBJECTIVE 1: Solving Trigonometric Equations That Are Linear in Form

Steps for Solving Trigonometric Equations that are Linear in Form

Step 1: Isolate the trigonometric function on one side of the equation.

Step 2: Determine the quadrants in which the terminal side of the argument of the function lies or determine the axis on which the terminal side of the argument of the function lies.

Step 3: **If the terminal side of the argument of the function lies within a quadrant,** then determine the reference angle and the value(s) of the argument on the interval $[0, 2\pi)$.
If the terminal side of the argument of the function lies along an axis, then determine the angle associated with it on the interval $[0, 2\pi)$ choosing from $0, \frac{\pi}{2}, \pi, \text{or } \frac{3\pi}{2}$.

Step 4: Use the period of the given function to determine the solutions.

EXAMPLES: Determine a general formula (or formulas) for the solution to each equation. Then determine the specific solutions (if any) on the interval $[0,2\pi)$.

8.5.2 $\tan\theta + \sqrt{3} = 0$ $\Rightarrow$ $\tan\theta = -\sqrt{3}$ ← Quadrant II or IV

$\boxed{\theta = \frac{2\pi}{3}, \frac{5\pi}{3}}$

8.5.4 $5\csc\theta - 3 = 2$

$\csc\theta = \frac{5}{5} \rightarrow \csc\theta = 1 \Rightarrow \sin\theta = 1 \Rightarrow \boxed{\theta = \pi/2}$

8.5.7 $2\cos 3\theta + \sqrt{2} = 0$

$\cos 3\theta = \frac{\sqrt{2}}{2}$

$3\theta = \frac{\pi}{4}, \frac{7\pi}{4}$

$\boxed{\theta = \frac{\pi}{12}, \frac{7\pi}{12}}$

8.5.9 $5\sec\left(\frac{3\theta}{2}\right) + 10 = 0$

We go to 3π since $u = \frac{3\theta}{2}$ $\to$ think: $\frac{3}{2}(2\pi) = 3\pi$

$\sec\left(\frac{3\theta}{2}\right) = -2 \Rightarrow u = \frac{3\theta}{2}, \quad \sec(u) = -2 \Rightarrow \cos(u) = -\frac{1}{2}$

Quadrant II, III

$u = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$

$\frac{3\theta}{2} = \frac{2\pi}{3} \to \boxed{\theta = \frac{4\pi}{9}} \qquad \frac{3\theta}{2} = \frac{4\pi}{3} \to \boxed{\theta = \frac{8\pi}{9}}$

OBJECTIVE 2: Solving Trigonometric Equations That Are Quadratic in Form

A quadratic equation has the form $ax^2 + bx + c,\ a \neq 0$. These equations are relatively straightforward to solve because we know several methods for solving these types of equations. If $f(\theta)$ is a trigonometric function, then the equation $(f(\theta))^2 + b(f(\theta)) + c,\ a \neq 0$ is said to be quadratic in form because we can transform it into a quadratic equation using the substitution $u = f(\theta)$.

EXAMPLES: Determine a general formula (or formulas) for the solution to each equation. Then determine the specific solutions (if any) on the interval $[0, 2\pi)$.

8.5.12 $2\cos^2\theta - 9\cos\theta - 5 = 0$

$u = \cos\theta$

$2u^2 - 9u - 5 = 0$

$s + t = -9$
$st = -10$

$2u^2 - 10u + 1u - 5 = 0$

$\to 2u(u-5) + (u-5) = 0$

$(2u+1)(u-5) = 0$

$u = -\frac{1}{2}, 5$

$\cos\theta = 5$ No solution!

$\cos\theta = -\frac{1}{2} \Rightarrow \boxed{\theta = \frac{2\pi}{3}, \frac{4\pi}{3}}$

8.5.14 $2\sin^2\left(\frac{\theta}{3} - \frac{\pi}{4}\right) - 1 = 0$

$u = \sin\left(\frac{\theta}{3} - \frac{\pi}{4}\right)$

$2u^2 - 1 = 0$

$u^2 = \frac{1}{2}$

$u = \pm\frac{1}{\sqrt{2}}$

$v = \frac{\theta}{3} - \frac{\pi}{4}$

$\sin(v) = \pm\frac{1}{\sqrt{2}}$

$v = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

① $\frac{\theta}{3} - \frac{\pi}{4} = \frac{\pi}{4} \Rightarrow \theta = \boxed{\frac{3\pi}{2}}$

② $\frac{\theta}{3} - \frac{\pi}{4} = \frac{3\pi}{4}$

$\theta = \frac{9\pi}{2}$ Not in $[0, 2\pi)$

③ $\frac{\theta}{3} - \frac{\pi}{4} = \frac{5\pi}{4}$

$\theta = \frac{15\pi}{2}$ Not in $[0, 2\pi)$

④ $\frac{\theta}{3} - \frac{\pi}{4} = \frac{7\pi}{4}$

$\theta = \frac{21\pi}{2}$ Not in $[0, 2\pi)$

OBJECTIVE 3: Solving Trigonometric Equations Using Identities

Determine a general formula (or formulas) for the solution to each equation. Then determine the specific solutions (if any) on the interval $[0,2\pi)$.

8.5.18 $\cos 2\theta + 3 = 5\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta \Rightarrow \cos^2\theta - 1 + \cos^2\theta = 2\cos^2\theta - 1$

Since we want only cosine, we use: $\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$2\cos^2\theta - 1 + 3 = 5\cos\theta$

$2\cos^2\theta + 2 - 5\cos\theta = 0$

$u = \cos\theta$

$2u^2 - 5u + 2 = 0$

$(2u-1)(u-2) = 0$

$\cos\theta = \frac{1}{2}$ $\quad \cos\theta = 2$ Not possible

$\boxed{\theta = \frac{\pi}{3}, \frac{5\pi}{3}}$

8.5.20 (CAUTION- be careful not to divide by an expression containing a variable)

$\sin 2\theta \sin\theta = \cos\theta$

$\hookrightarrow \sin 2\theta = 2\sin\theta\cos\theta$

$2\sin^2\theta\cos\theta = \cos\theta$

$2(2\cos^2\theta - 1)\cos\theta = \cos\theta$

$4\cos^3\theta - 2\cos\theta = \cos\theta$

$4\cos^3\theta - 3\cos\theta = 0$

$\cos\theta(4\cos^2\theta - 1) = 0$

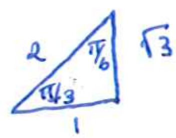
$\cos\theta = 0$

$\boxed{\theta = \frac{\pi}{2}, \frac{3\pi}{2}}$

$\cos^2\theta = \frac{1}{4}$

$\cos\theta = \pm\frac{1}{2}$

$\boxed{\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}}$



8.5.22 $4\cos\theta = 3\sec\theta$

$4\cos^2\theta = 3$

$\cos^2\theta = 3/4$

$\cos\theta = \pm\sqrt{3}/2$

$\boxed{\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$

OBJECTIVE 4: Solving Other Types of Trigonometric Equations

8.5.26 $3\sin\theta = -3\cos\theta$

$\tan\theta = -1$ Quadrant II, IV

$\boxed{\theta = \frac{3\pi}{4}, \frac{7\pi}{4}}$

8.5.27 $\sin\theta + \cos\theta = -\sqrt{2}$

$(\sin\theta + \cos\theta)^2 = (-\sqrt{2})^2$

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 2$

$1 + 2\sin\theta\cos\theta = 2$

$2\sin\theta\cos\theta = 1$

$\sin 2\theta = 1$

$2\theta = \pi/2, \frac{5\pi}{2}$

since we're looking at 2θ, we should consider $[0, 4\pi)$

think $2(2\pi) = 4\pi$

$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

does not work!

This is an erroneous solution. We get these when we square both sides.

$\boxed{\theta = \frac{5\pi}{4}}$

8.5.29 $\sin^2\theta + \sqrt{3}\sin\theta\cos\theta = 0$