HW 44 Solutions:

Section 8.2 The Sum and difference Formulas

Unlike the identities in the previous section, where our identities contained only one variable, many of the identities in this section contain two variables.

OBJECTIVE 1: Understanding the Sum and Difference Formulas for the Cosine Function

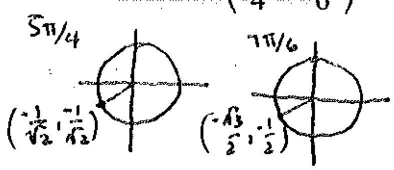
The Sum and Difference Formulas for the Cosine Function

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$	**Cosine of the Sum of Two Angles Formula**
$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$	**Cosine of the Difference of Two Angles Formula**

In Class. Prove the Cosine of the Difference of Two Angles Formula.

EXAMPLES. Find the exact value of each trig expression without the use of a calculator.

8.2.7 $\cos\left(\frac{5\pi}{4}-\frac{7\pi}{6}\right) = \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{7\pi}{6}\right)+\sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{7\pi}{6}\right)$



$$= \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{\sqrt{3}}{2}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}} = \boxed{\frac{1+\sqrt{3}}{2\sqrt{2}}}$$

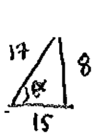
8.2.13 $\sec\left(-\frac{\pi}{12}\right) = \sec\left(\frac{\pi}{4}-\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right)} = \frac{1}{\cos(\pi/4)\cos(\pi/3)+\sin(\pi/4)\sin(\pi/3)}$

$$= \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\frac{1+\sqrt{3}}{2\sqrt{2}}} = \boxed{\frac{2\sqrt{2}}{1+\sqrt{3}}}$$

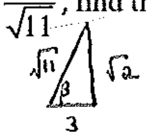
8.2.14 Suppose the terminal side of angle α lies in Quadrant I and the terminal side of angle β lies in Quadrant IV. If $\sin\alpha=\frac{8}{17}$ and $\cos\beta=\frac{3}{\sqrt{11}}$, find the exact value of $\cos(\alpha+\beta)$.

$\sin\alpha=\frac{8}{17}$

$\cos\alpha=\frac{15}{17}$



$\cos\beta=\frac{3}{\sqrt{11}}$



$\sin\beta=-\frac{\sqrt{2}}{\sqrt{11}}$

Quadrant IV means sine is negative

$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$$=\left(\frac{15}{17}\right)\left(\frac{3}{\sqrt{11}}\right)-\left(\frac{8}{17}\right)\left(\frac{-\sqrt{2}}{\sqrt{11}}\right)$$

$$=\boxed{\frac{45+8\sqrt{2}}{17\sqrt{11}}}$$

OBJECTIVE 2: Understanding the Sum and Difference Formulas for the Sine Function

Recall the Cofunction Identities for Sine and Cosine (6.4).

1.

2.

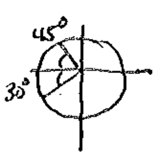
The Sum and Difference Formulas for the Sine Function

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$	**Sine of the Sum of Two Angles Formula**
$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$	**Sine of the Difference of Two Angles Formula**

In Class: Prove the sine of the sum of two angles formula by using a cofunction identity and the cosine of the difference of two angles formula.

EXAMPLES. Find the exact value of each trig expression without the use of a calculator.

8.2.21 sin(210°+135°) $= \sin(210)\cos(135)+\cos(210)\sin(135)$



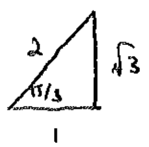
$$= \left(\frac{-1}{2}\right)\left(\frac{-1}{\sqrt{2}}\right)+\left(\frac{-\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \boxed{\frac{1-\sqrt{3}}{2\sqrt{2}}}$$

8.2.24 $\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{2\pi}{3}\right)+\cos\left(\frac{\pi}{12}\right)\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{12}+\frac{2\pi}{3}\right) = \sin\left(\frac{9\pi}{12}\right) = \sin\left(\frac{3\pi}{4}\right) = \boxed{\frac{1}{\sqrt{2}}}$

EXAMPLES. Find the exact value of each trig expression without the use of a calculator.

8.2.40 $\dfrac{\tan\left(\frac{4\pi}{5}\right)-\tan\left(\frac{2\pi}{15}\right)}{1+\tan\left(\frac{4\pi}{5}\right)\tan\left(\frac{2\pi}{15}\right)} = \tan\left(\frac{4\pi}{5}-\frac{2\pi}{15}\right) = \tan\left(\frac{10\pi}{15}\right) = \tan\left(\frac{2\pi}{3}\right) = \boxed{-\sqrt{3}}$



8.2.41 $\tan(15°) = \tan(45° - 30°) = \dfrac{\tan(45°)-\tan(30°)}{1+\tan(45°)\tan(30°)} = \dfrac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}}$

$= \boxed{\dfrac{\sqrt{3}-1}{\sqrt{3}+1}}$

8.2.44 Suppose the terminal side of angle α lies in Quadrant II and the terminal side of angle β lies in Quadrant III. If $\sin\alpha = \frac{1}{4}$ and $\cos\beta = -\frac{3}{5}$, find the exact value of $\tan(\alpha+\beta)$.

OBJECTIVE 4: Using the Sum and Difference Formulas to Verify Identities

If one side of an identity includes an expression of the form $\cos(\alpha \pm \beta)$ or $\sin(\alpha \pm \beta)$ or $\tan(\alpha \pm \beta)$, first use one of the sum or difference formulas, then use the strategies from Section 8.1 to verify identities.

8.2.46 $\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$

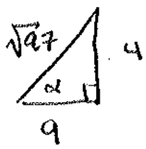
$\hookrightarrow \sin\left(\frac{\pi}{2}+\theta\right) = \sin\left(\frac{\pi}{2}\right)\cos\theta + \cos\left(\frac{\pi}{2}\right)\sin\theta$

$= \cos\theta$ ✓

means Quadrant III

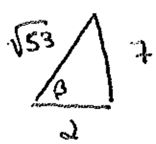
8.2.30 Suppose that α is an angle such that $\tan\alpha = \frac{4}{9}$ and $\sin\alpha < 0$. Also suppose that β is an angle such that $\cot\beta = -\frac{2}{7}$ and $\sec\beta > 0$. Find the exact value of $\sin(\alpha+\beta)$.

Quadrant IV



$\sin\alpha = \frac{-4}{\sqrt{97}}$

$\cos\alpha = \frac{-3}{\sqrt{97}}$



$\sin(\beta) = -\frac{7}{\sqrt{53}}$

$\cos(\beta) = \frac{2}{\sqrt{53}}$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{-4}{\sqrt{97}}\right)\left(\frac{2}{\sqrt{53}}\right) + \left(\frac{-3}{\sqrt{97}}\right)\left(\frac{-7}{\sqrt{53}}\right)$

$= \frac{-8+21}{\sqrt{5141}} = \boxed{\frac{13}{\sqrt{5141}}}$

OBJECTIVE 3: Understanding the Sum and Difference Formulas for the Tangent Function

The Sum and Difference Formulas for the Tangent Function

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ **Tangent of the Sum of Two Angles Formula**

$\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$ **Tangent of the Difference of Two Angles Formula**

Derive (or Prove) the tangent of the sum of two angles formula.

8.2.51 $\csc(\alpha-\beta)=\dfrac{\csc\alpha\csc\beta}{\cot\beta-\cot\alpha}$

$$\hookrightarrow \csc(\alpha-\beta) = \frac{1}{\sin(\alpha-\beta)} = \frac{1}{\sin\alpha\cos\beta - \cos\alpha\sin\beta}\cdot\frac{\frac{1}{\sin\alpha\sin\beta}}{\frac{1}{\sin\alpha\sin\beta}} = \frac{\csc\alpha\ \csc\beta}{\cot\beta - \cot\alpha} \quad \checkmark$$

OBJECTIVE 5: Using the sum and Difference Formulas to Evaluate Expressions Involving Inverse Trigonometric Functions

Review Inverse Trigonometric Functions (7.4)

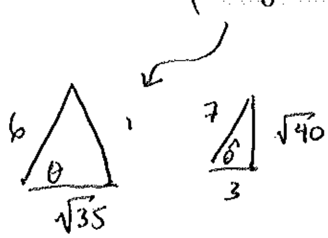
$y=\sin^{-1}x$ $\qquad$ $y=\cos^{-1}x$ $\qquad$ $y=\tan^{-1}x$

EXAMPLES. Find the exact value of each expression without the use of a calculator.

8.2.52 $\sin\left(\tan^{-1}1+\cos^{-1}\dfrac{\sqrt{3}}{2}\right)$ $= \sin\left(\tan^{-1}1\right)\cos\left(\cos^{-1}\frac{\sqrt{3}}{2}\right) + \cos\left(\tan^{-1}1\right)\sin\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\sin\left(\frac{1}{2}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}$$

8.2.54 $\sin\left(\sin^{-1}\dfrac{1}{6}-\cos^{-1}\left(-\dfrac{3}{7}\right)\right)$ $= \sin\left(\sin^{-1}\left(\frac{1}{6}\right)\right)\cos\left(\cos^{-1}\left(-\frac{3}{7}\right)\right) - \cos\left(\sin^{-1}\left(\frac{1}{6}\right)\right)\sin\left(\cos^{-1}\left(-\frac{3}{7}\right)\right)$



$$= \left(\frac{1}{6}\right)\left(-\frac{3}{7}\right) - \left(\frac{\sqrt{35}}{6}\right)\left(\frac{\sqrt{40}}{7}\right) = \frac{-3-\sqrt{1400}}{42}$$

$$= \boxed{\frac{-3-10\sqrt{14}}{42}}$$