HW 45 Solutions :

## Section 8.3 The Double-Angle and Half-Angle Formulas

OBJECTIVE 1: Understanding the Double-Angle Formulas

**Double-Angle Formulas**

$\sin 2\theta = 2\sin\theta\cos\theta$ $\quad$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\quad$ $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

In Class: Use the sum and difference formulas to prove the double-angle formula for $\cos 2\theta$.

Write the two additional forms for $\cos 2\theta$.

Show the derivation for $\cos 2\theta$ in terms of sine. (If you memorize only the formula given above for $\cos 2\theta$, you can easily derive the forms in terms of just sine, or just cosine.)

EXAMPLES. Rewrite each expression as the sine, cosine or tangent of a double-angle. Then find the exact value of the trigonometric expression without the use of a calculator.

8.3.1 $2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) = \sin\left(2\cdot\frac{\pi}{8}\right) = \sin\left(\frac{\pi}{4}\right) = \boxed{\frac{1}{\sqrt{2}}}$

8.3.9 $1-2\sin^2(-157.5°) = 1-\sin^2(-157.5°) - \sin^2(-157.5°) = \cos^2(-157.5°) - \sin^2(-157.5°)$

$= \cos\left(2(-157.5)\right) = \cos(-315°) = \cos(315°) = \boxed{\frac{1}{\sqrt{2}}}$

8.3.14 Given $\cot\theta = 10$ and the terminal side of $\theta$ lies in Quadrant III, determine $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.


$\sin(2\theta) = 2\sin\theta\cos\theta$

$= 2\left(\frac{-1}{\sqrt{101}}\right)\left(\frac{-10}{\sqrt{101}}\right) = \boxed{\frac{20}{101}}$

$\tan 2\theta = \boxed{\frac{20}{99}}$

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= \left(\frac{-10}{\sqrt{101}}\right)^2 - \left(\frac{-1}{\sqrt{101}}\right)^2 = \frac{100}{101} - \frac{1}{101} = \boxed{\frac{99}{101}}$

8.3.16 Given the information $\cos 2\theta = \frac{3}{5}; \frac{3\pi}{2} < 2\theta < 2\pi$, determine the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$.

**OBJECTIVE 2: Understanding the Power Reduction Formulas**

In calculus it is often helpful to *reduce* even powered trigonometric expressions so that the trigonometric function is written to a power of 1.

**The Power Reduction Formulas**

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

8.3.21 Rewrite the given function $f(x) = 2\sin^4 x$ as an equivalent function containing only cosine terms raised to a power of 1.

**OBJECTIVE 3: Understanding the Half-Angle Formulas**

The half-angle formulas can be derived from the Power Reduction Formulas and taking the square root of both sides of the equation. The choice of which root (positive or negative) depends on the quadrant in which the terminal side of $\theta$ lies.

**The Half-Angle Formulas for Sine and Cosine**

$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos\alpha}{2}}$ for $\frac{\alpha}{2}$ in Quadrant I or Quadrant II.

$\sin\left(\frac{\alpha}{2}\right) = -\sqrt{\frac{1-\cos\alpha}{2}}$ for $\frac{\alpha}{2}$ in Quadrant III or Quadrant IV.

$\cos\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1+\cos\alpha}{2}}$ for $\frac{\alpha}{2}$ in Quadrant I or Quadrant IV.

$\cos\left(\frac{\alpha}{2}\right) = -\sqrt{\frac{1+\cos\alpha}{2}}$ for $\frac{\alpha}{2}$ in Quadrant II or Quadrant III.

OBJECTIVE 4: Using the Double-Angle, Power Reduction, and Half-Angle Formulas to Verify Identities

If one side of an identity includes a trigonometric expression involving $2\theta$ or $\frac{\theta}{2}$, first substitute one of the formulas from this section, then use strategies developed in Section 8.1 for verifying identities.

8.3.39 Verify the identity.

$$\cot 2\theta = \frac{1}{2}\sec\theta\csc\theta - \tan\theta$$

OBJECTIVE 5: Using the Double-Angle, Power Reduction, and Half-Angle Formulas to Evaluate Expressions Involving Inverse Trigonometric Functions

EXAMPLES. Find the exact value of each expression without the use of a calculator.

8.3.46 $\cos\left(2\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right) = \cos^2\left(\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right) - \sin^2\left(\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right)\right)$

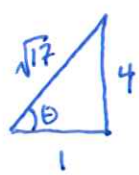
$$= \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{-1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2} - \frac{1}{2} = \boxed{0}$$

8.3.52 $\cos\left(\frac{1}{2}\tan^{-1}4\right) = \sqrt{\frac{1+\cos(\tan^{-1}(4))}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{17}}}{2}} = \sqrt{\frac{\sqrt{17}+1}{2\sqrt{17}}}$

↑
Quadrant I
(can't be Q IV)

$$= \boxed{\sqrt{\frac{17+\sqrt{17}}{34}}}$$



## The Half-Angle Formulas for Tangent

$$\tan\left(\frac{\alpha}{2}\right) = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \quad \text{for } \frac{\alpha}{2} \text{ in Quadrant I or Quadrant III.}$$

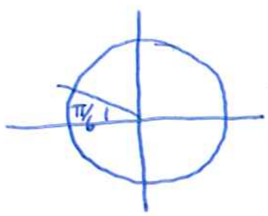
$$\tan\left(\frac{\alpha}{2}\right) = -\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \quad \text{for } \frac{\alpha}{2} \text{ in Quadrant II or Quadrant IV.}$$

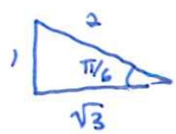
$$\tan\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{\sin\alpha} \quad \text{for } \frac{\alpha}{2} \text{ in any quadrant.}$$

$$\tan\left(\frac{\alpha}{2}\right) = \frac{\sin\alpha}{1+\cos\alpha} \quad \text{for } \frac{\alpha}{2} \text{ in any quadrant.}$$

8.3.29 Use a half-angle formula to evaluate the expression without using a calculator.

$$\tan\left(\frac{5\pi}{12}\right) = \tan\left(\frac{1}{2}\cdot\frac{5\pi}{6}\right) = \frac{\sin(5\pi/6)}{1+\cos(5\pi/6)} = \frac{1/2}{1-\sqrt{3}/2} = \boxed{\frac{1}{2-\sqrt{3}}}$$




8.3.35 Use the given information $\sec\alpha = -\frac{5}{4}; \pi < \alpha < \frac{3\pi}{2}$ to determine the values of $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$.