Section 4.6 Rational Functions and Their Graphs

> **Definition Rational Function**
>
> A rational function is a function of the form $f(x) = \dfrac{g(x)}{h(x)}$ where g and h are polynomial functions such that $h(x) \neq 0$.

Objective 1: Finding the Domain and Intercepts of Rational Functions

Rational functions are defined for all values of x **except** those for which the denominator $h(x)$ is equal to zero.

If $f(x)$ has a y-intercept, it can be found by evaluating $f(0)$ provided that $f(0)$ is defined.

If $f(x)$ has any x-intercepts, they can be found by solving the equation $g(x) = 0$ (provided that g and h do not share a common factor).

4.6.1 and 5

For the given rational function, determine the following:

1. Function $f(x) = \dfrac{x}{x^2 - 1}$

a) the domain

b) the y-intercept (if any)

c) the x-intercepts

5. Function $f(x) = \dfrac{x^2 + x - 12}{x^2 + x}$

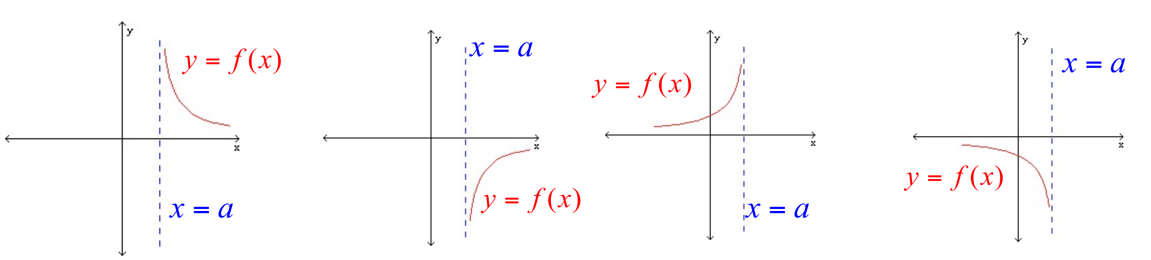
a) the domain

b) the y-intercept (if any)

c) the x-intercepts

Objective 2: Identifying Vertical Asymptotes

> **Definition Vertical Asymptote**
>
> The vertical line $x = a$ is a **vertical asymptote** of a function $y = f(x)$ if *at least* one of the following occurs:



$f(x) \to \infty$ as $x \to a^+$ $f(x) \to -\infty$ as $x \to a^+$ $f(x) \to \infty$ as $x \to a^-$ $f(x) \to -\infty$ as $x \to a^-$

A rational function of the form $f(x) = \dfrac{g(x)}{h(x)}$ where $g(x)$ and $h(x)$ have no common factors will have a vertical asymptote at $x = a$ if $h(a) = 0$.

It is essential to cancel any common factors before locating the vertical asymptotes.

If there is an *x*-intercept near the vertical asymptote, it is essential to choose a test value that is between the *x*-intercept and the vertical asymptote.

4.6.7 and 10

Find all vertical asymptotes and create a rough sketch of the graph near each asymptote.

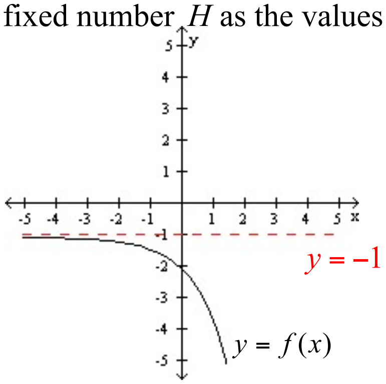
7. $f(x) = \dfrac{3}{x+1}$. The vertical asymptote is $x =$ ______.

10. $f(x) = \dfrac{x+3}{x^2 - 6x + 8}$. The vertical asymptote is $x =$ ______.

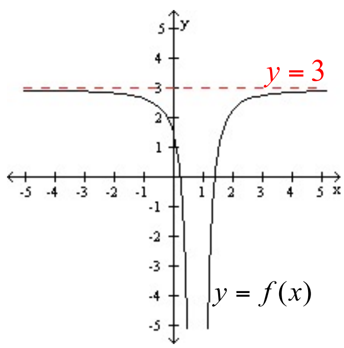
Objective 3: Identifying Horizontal Asymptotes

Definition Horizontal Asymptote

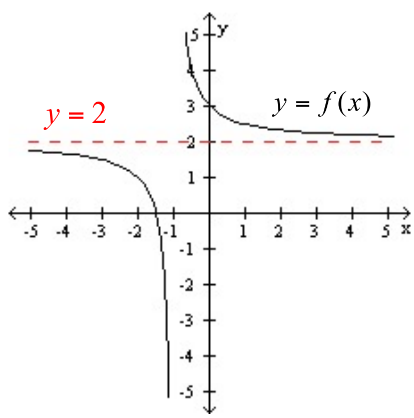
A horizontal line $y = H$ is a **horizontal asymptote** of a function f if the values of $f(x)$ approach some fixed number H as the values of x approach ∞ or $-\infty$.



The line $y = -1$ is a horizontal asymptote because the values of $f(x)$ approach -1 as x approaches $-\infty$.



The line $y = 3$ is a horizontal asymptote because the values of $f(x)$ approach 3 as x approaches $\pm\infty$.



The line $y = 2$ is a horizontal asymptote because the values of $f(x)$ approach 2 as x approaches $\pm\infty$.

Properties of Horizontal Asymptotes of Rational Functions

- Although a rational function can have many vertical asymptotes, it can have at most one horizontal asymptote.
- The graph of a rational function will never intersect a vertical asymptote but may intersect a horizontal asymptote.
- A rational function $f(x) = \frac{g(x)}{h(x)}$ that is written in lowest terms (all common factors of the numerator and denominator have been cancelled) will have a horizontal asymptote whenever the degree of $h(x)$ is greater than or equal to the degree of $g(x)$.

Finding Horizontal Asymptotes of a Rational Function

Let $f(x) = \frac{g(x)}{h(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + \cdots + b_1 x + b_0}$, $a_n \neq 0, b_m \neq 0$

where f is written in lowest terms, n is the degree of g, and m is the degree of h.

- If $m > n$, then $y = 0$ is the horizontal asymptote.
- If $m = n$, then the horizontal asymptote is $y = \frac{a_n}{b_m}$, the ratio of the leading coefficients.
- If $m < n$, then there are no horizontal asymptotes.

4.6.13, 16, 17

Find the equation of all horizontal asymptotes (if any) of the rational function.

13. $f(x) = \frac{3}{x+1}$. There is a horizontal asymptote at $y =$ _______.

16. $f(x) = \frac{2x^3 - x^2 + 1}{x^2 + 1}$. There is a horizontal asymptote at $y =$ _______.

17. $f(x) = \frac{3x^2 - 4x^3}{7x^3 + x^2 - x + 1}$. There is a horizontal asymptote at $y =$ _______.