Section 4.4 Synthetic Division; The Remainder and Factor Theorems

Objective 1: Using the Division Algorithm

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomial functions with $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomial functions $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

or equivalently $$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

where the remainder $r(x) = 0$ or is of degree less than the degree of $d(x)$.

Corollary to the Division Algorithm

If a polynomial $f(x)$ of degree greater than or equal to one is divided by a polynomial $d(x)$ where $d(x)$ is of degree one, then there exists a unique polynomial function $q(x)$ and a constant r such that

$$f(x) = d(x)q(x) + r.$$

Example: Divide 127 by 32. Can you see the analogous operation with whole numbers?

Objective 2: Using Synthetic Division

If a polynomial $f(x)$ is divided by $d(x) = x - c$, then we can use a "shortcut method" called **synthetic division** to find the quotient, $q(x)$, and the remainder, r.

You can see how much more compact the synthetic division process is compared to long division.

Step 1: The constant coefficient, *c*, of the divisor $d(x) = x - c$ is written to the far left while all coefficients of the polynomial $f(x)$ are written inside the symbol $\underline{\left|\quad\quad\quad\right.}$. Once again, be sure to include the 0 for $0x^2$.

This is c in $x - c$. →

$$\begin{array}{r|rrrrr} 2 & 4 & -3 & 0 & 5 & -6 \\ \hline \end{array}$$

These are the coefficients of $f(x) = 4x^4 - 3x^3 + 5x - 6$.

Step 2: Bring down the leading coefficient 4

$$\begin{array}{r|rrrrr} 2 & 4 & -3 & 0 & 5 & -6 \\ \hline & \downarrow & & & & \\ \hline & 4 & & & & \end{array}$$

Step 3: Multiply *c* times the leading coefficient that was just brought down. In this case we multiply $2 \cdot 4$. The product (8 in this case) is written in the next column in the second row.

Multiply $2 \cdot 4 = 8$

$$\begin{array}{r|rrrrr} 2 & 4 & -3 & 0 & 5 & -6 \\ \hline & \downarrow & 8 & & & \\ \hline & 4 & & & & \end{array}$$

Step 4: Add down the column and write the sum (5 in this case) in the bottom row.

$$\begin{array}{r|rrrrr} 2 & 4 & -3 & 0 & 5 & -6 \\ \hline & & 8 & & & \\ \hline & 4 & 5 & & & \end{array}$$

We now repeat this process multiplying $c = 2$ times the value in the last row, always adding down the columns.

Mulitiply $2 \cdot 5 = 10$

$$\begin{array}{r|rrrrr} 2 & 4 & -3 & 0 & 5 & -6 \\ \hline & & 8 & 10 & & \\ \hline & 4 & 5 & & & \end{array}$$

Add 0+10=10

$$\begin{array}{r|rrrrr} 2 & 4 & -3 & 0 & 5 & -6 \\ \hline & & 8 & 10 & & \\ \hline & 4 & 5 & 10 & & \end{array}$$

Mulitiply $2 \cdot 10 = 20$

$$\begin{array}{r|rrrrr} 2 & 4 & -3 & 0 & 5 & -6 \\ \hline & & 8 & 10 & 20 & \\ \hline & 4 & 5 & 10 & & \end{array}$$

Add 5+20=25

$$\begin{array}{r|rrrrr} 2 & 4 & -3 & 0 & 5 & -6 \\ \hline & & 8 & 10 & 20 & \\ \hline & 4 & 5 & 10 & 25 & \end{array}$$

Mulitiply $2 \cdot 25 = 50$

$$\begin{array}{r|rrrrr} 2 & 4 & -3 & 0 & 5 & -6 \\ \hline & & 8 & 10 & 20 & 50 \\ \hline & 4 & 5 & 10 & 25 & \end{array}$$

Add -6+50=44

$$\begin{array}{r|rrrrr} 2 & 4 & -3 & 0 & 5 & -6 \\ \hline & & 8 & 10 & 20 & 50 \\ \hline & 4 & 5 & 10 & 25 & 44 \end{array}$$

The last row now represents the quotient and the remainder. The last entry of the bottom row (44 in this case) is the remainder. The other numbers of the last row represent the coefficients of the quotient.

$$\begin{array}{r|rrrrr} 2 & 4 & -3 & 0 & 5 & -6 \\ \hline & & 8 & 10 & 20 & 50 \\ \hline & 4 & 5 & 10 & 25 & 44 \end{array}$$

remainder $r = 44$

coefficients of $q(x) = 4x^3 + 5x^2 + 10x + 25$

Note that synthetic division can only be used when the divisor, $d(x)$ has the form $(x-c)$.

4.4.6 Use synthetic division to divide $f(x)$ by $x - c$ then write $f(x)$ in the form $f(x) = (x - c)q(x) + r$.

$f(x) = -2x^4 + 3x^3 - x^2 - 2$; $x+1$

Objective 3: Using the Remainder Theorem

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.

4.4.14 Use synthetic division and the remainder theorem to find the remainder when $f(x)$ is divided by $x - c$.

$f(x) = -4x^3 + 2x - 1$; $x+3$

Objective 4: Using the Factor Theorem

The Factor Theorem

The polynomial $x-c$ is a factor of the polynomial $f(x)$ if and only if $f(c)=0$.

4.4.21 Use synthetic division and the factor theorem to determine if $x - c$ is a factor of $f(x)$.

$f(x) = 2x^3 + 5x^2 - 8x - 5$; $x - \frac{1}{2}$

Objective 5: Sketching the Graph of a Polynomial Function

The polynomial must be written in completely factored form. Use the four-step process in Section 4.3 to sketch the polynomial.

4.4.33 Find the remaining zeros of $f(x)$ given that c is a zero. Then rewrite $f(x)$ in completely factored form and sketch its graph.

$f(x) = 4x^4 + 5x^3 - 3x^2 - 5x - 1$; $c = -1$ is a zero of multiplicity 2