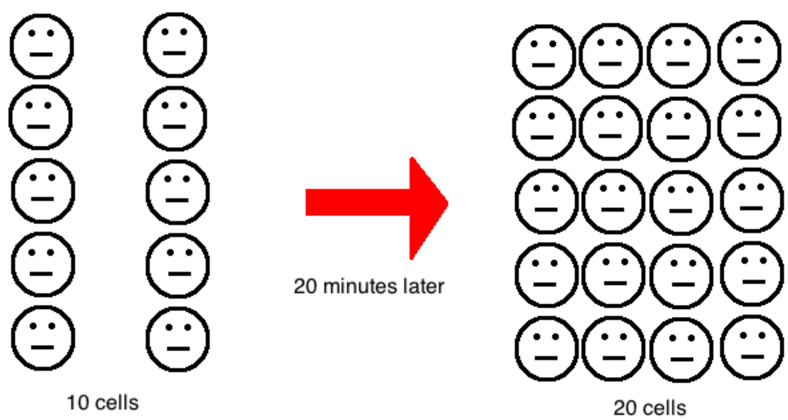
8. Lecture 8

Objectives

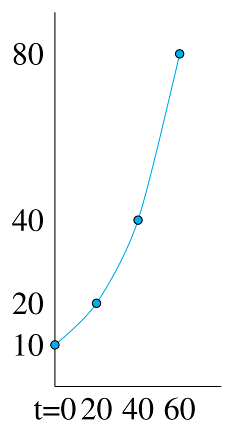
- ☐ I understand that two measurements can have a nonlinear relationship.
- ☐ I know the shapes of linear, exponential, logarithmic, logistic, and polynomial functions (up to degree 4).
- ☐ I can fit data with any of the above function-types using a graphic calculator.
- ☐ I can find the residuals to help me determine if a particular function type is appropriate.

Up until now, we have focused on linear relationships, many interesting relationships in nature are not linear. Most populations do not grow linearly.

Why? Well, let's consider the very simple model of bacteria like E. coli. E. coli cells split into two every 20 minutes. So if we begin with 10 cells, we assume that after 20 minutes, we get double that amount: 40 cells.



While this may sound linear on the surface, let's see what happens after a few 20-minute intervals:



As you can see, this is not a line at all! This is an exponential function. In fact, exponential growth describes all unconstrained population growth.

Exponential Growth 8.1

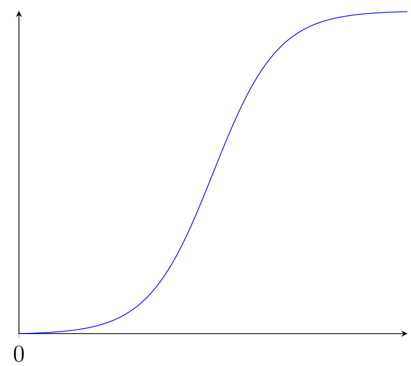
Biological populations, without constraints like limits on resources and space, grow according to some exponential function.

Exponential growth is of the form

$$y = ae^{rt}$$

where a is the initial population and r is the growth rate per time step.

In our model of E. coli growth, we don't consider death rates nor how those death rates change with respect to external limitations, like insufficient food or space. Growth begins exponential until those limitations are felt by the population. It then levels off and flattens out to the maximum population. Below is an image of a logistic graph.



A logistic function is of the form

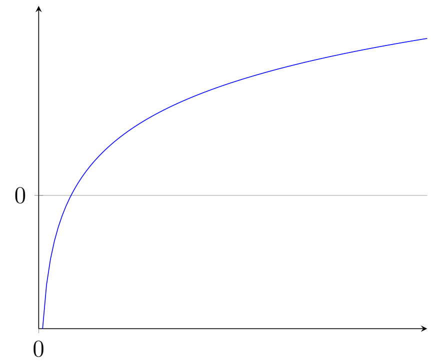
$$y = \frac{L}{1 + e^{-kx}}$$

where L is the initial population. We will revisit this function and why it is written this way during the differential equations portion of this class.

Logistical Growth 8.2

Biological populations with constraints (like limited food or space) grow according to some logistical function.

There are two other kinds of nonlinear functions the graphing calculator is equipped to handle. The first is logarithmic functions. Logarithmic functions are inverses of exponential functions. They are a class of functions that go to infinite very slowly. They appear as if they eventually become flat, but don't be fooled!



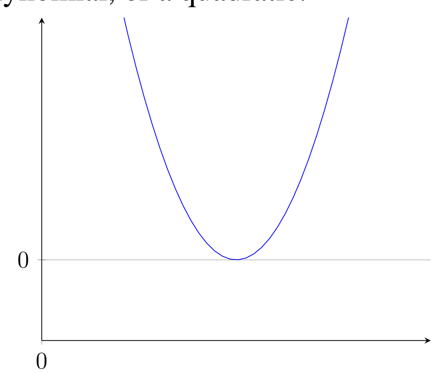
Logarithmic functions are of the form

$$y = a\ln(x) + b$$

Lastly, your calculator can fit polynomial functions. Polynomials are functions of the form

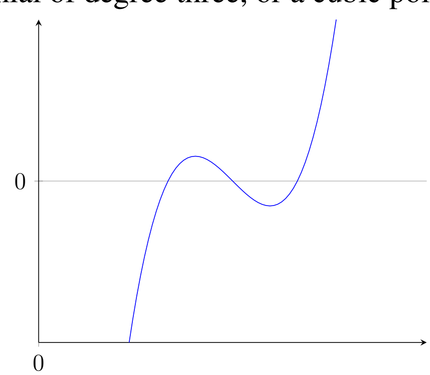
$$y = ax^n + bx^{n-1} + cx^{n-2} + \dots$$

where n is the degree of the polynomial and $a, b, c, \dots$ are constants. These are functions that change direction (e.g. positive slope to negative slope) $n-1$ times. For example, the function graphed below is a degree-2 polynomial, or a quadratic.



Notice that it changes direction 1 at the vertex of. Recall from calculus that these points where the direction changes are the critical points of the function, where the derivative is zero.

Here is example of a polynomial of degree three, or a cubic polynomial.



Again, notice that it changes direction twice, precisely at the critical point

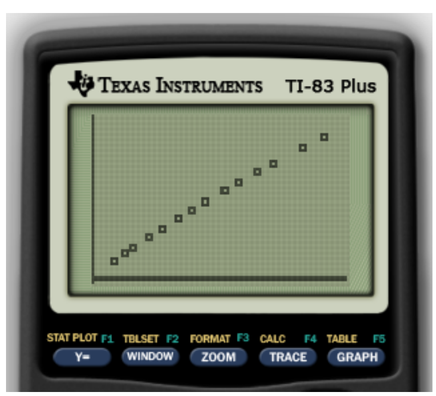
Because we can control how many times a polynomial can change direction by taking a higher degree, we can always fit data to a polynomial. While we can get a good fit, however, it does not mean that this fit is physically realistic. While we will not focus on it too much in this lecture, we will generally try to determine which function to use not just by the data collected but also by the physical situation.

8.0.1. *Best Fit Curves.* We now discuss how to use a graphing calculator to find a best fit curve. As you will see, it is the same process as finding the best fit line. The steps outlined here will skip some steps as they were covered in previous lectures. If you do not remember them, please check back on earlier notes.

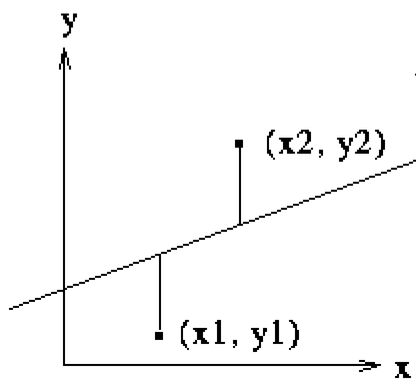
Let's begin with a sample data set. When plugging these values into your calculator, be sure to put the x values into column L_1 and the y values into column L_2.

y	x
10.07E0	77.6E0
14.73E0	114.9E0
17.94E0	141.1E0
23.93E0	190.8E0
29.61E0	239.9E0
35.18E0	289.0E0
40.02E0	332.8E0
44.82E0	378.4E0
50.76E0	434.8E0
55.05E0	477.3E0
61.01E0	536.8E0
66.40E0	593.1E0
75.47E0	689.1E0
81.78E0	760.0E0

Notice that when we plot the data, it appears linear. If you look at the functions described above, many of them can look linear if we only look at a small segment.



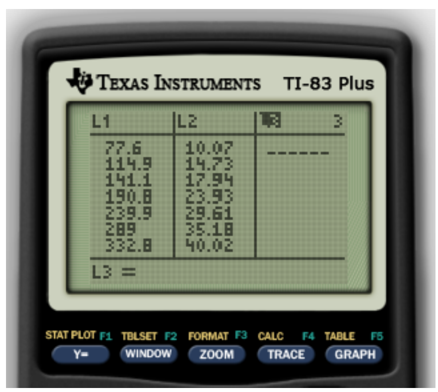
How do we distinguish a linear relationship against a nonlinear one? The answer is to look at the **residuals**. A residual is the difference (in the y-value) between a data point and the best fit curve.



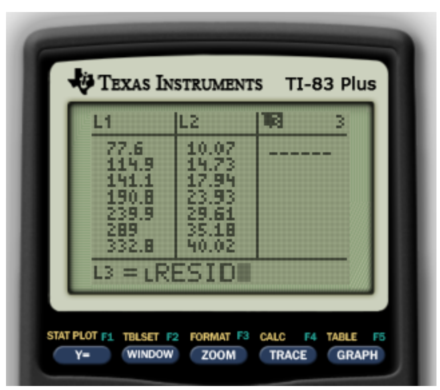
We use these distances to see if we can detect some curvature about the line. Here is how we graph residuals using your calculator.

- Find the best fit line: STAT → CALC → LinReg(ax + b)
- Make a list of residuals: STAT → EDIT → Edit...

Navigate to the L_3 column and navigate to the top.

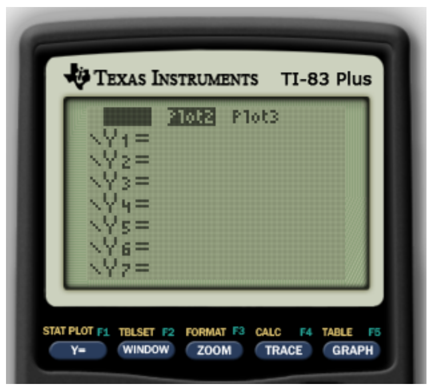


To access the residuals, press 2nd and STAT. Under the NAMES menu, select RESID.

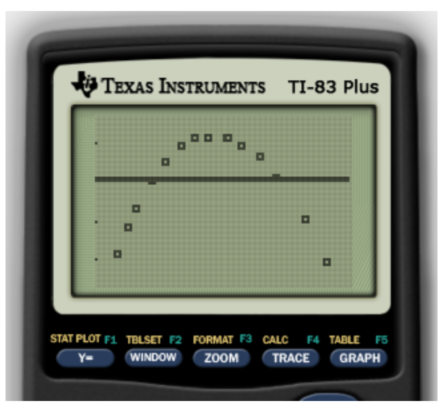


Press ENTER and the third column will fill with the residuals. To graph them, first quit by pressing 2nd and MODE. Navigate to the STAT PLOT menu and edit Plot2. Select ON and change the Ylist to L_3

Quit this menu. Then press Y=. At the top, you will see Plot1 and Plot2 highlighted. Navigate upward to Plot1 and hit enter to unhighlight it.



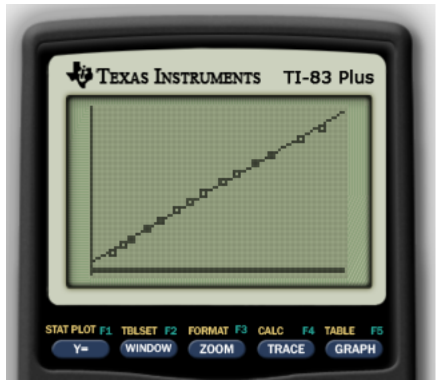
Then press ZOOM and select ZoomStat. Then you will see the residuals.



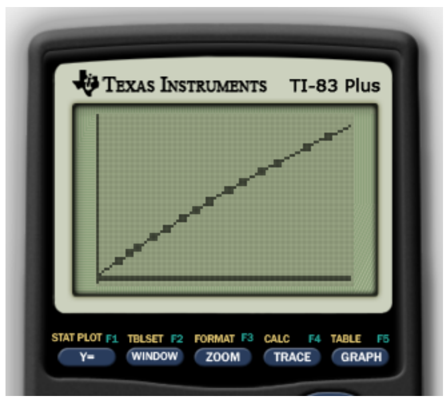
The graph of the residuals tells us that the data curves around the line, suggesting that a line is *not* appropriate for this data. Instead, we should try a nonlinear function.

Nonlinear best-fit curves can be graphed just like the lines. The functions available on your calculator are listed below. Each has the accompanying graph for the data set in this lecture.

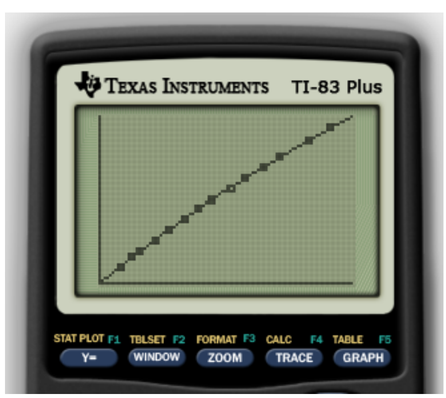
(1) `LinReg(ax+b)` for lines



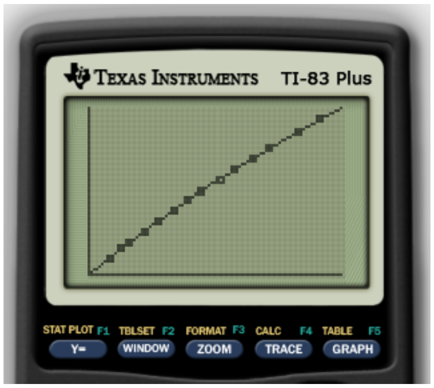
(2) `QuadReg` for polynomials of degree 2, i.e. quadratics



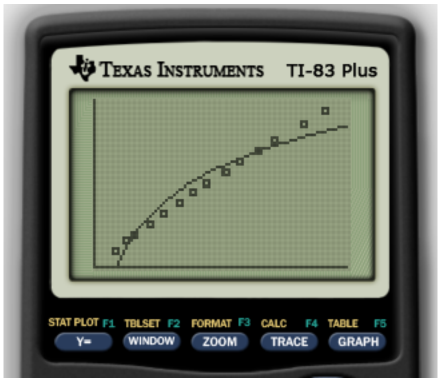
(3) `CubicReg` for polynomials of degree 3, i.e. cubics



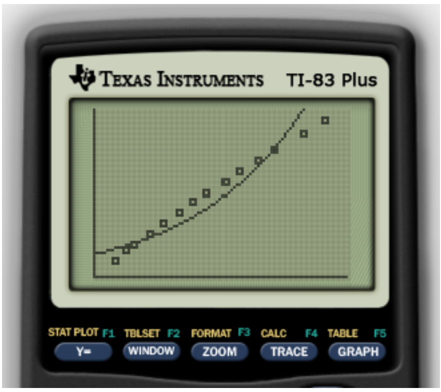
(4) `QuartReg` for polynomials of degree 4, i.e. quartics



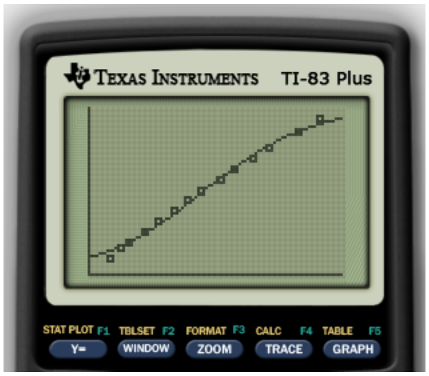
(5) `LnReg` for natural logs



(6) `ExpReg` for exponentials



`Logistic` for logistiscs



When comparing all the possible functions, the polynomials will generally fit the best (as you can see above). We generally want to pick the function based on *where* the data comes from rather than by just the graph.

One last calculator trick that may make testing out these various functions faster is the following. When you want to plot a best-fit curve, begin by pressing `Y=`. From there, press `VARS` and select `Statistics`. Navigate over to the `EQ` and select `RegEQ`. Then the last best-fit curve you calculated will appear in the `Y=` menu.

Summary of Ideas: Lecture 8

- Some pairs of measurements, like population and time, do not have a linear relationship.
- The nonlinear relationships we cover here are exponential, logarithmic, logistic, and polynomial.
- Populations grow exponentially unless there are constraints on resources. Then the growth is logistic.
- Polynomials can generally fit any kind of data very well, although they do not generally describe a lot of natural phenomenon.
- We use residuals to detect curvature of data around a best fit line.

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