7. LECTURE 7

Objectives

- ☐ I know that a vector can be filled with any kind of information.
- ☐ I can define a unit vector.
- ☐ I can calculate the dot product of two vectors.

In the last lecture, we thought of vectors as being descriptions in space. In reality, a vector can contain any kinds of data. They simply summarize what is happening in a system.

In the context of physics, most systems can be summarized by position and possibly time. So a common vector might be

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

where t represents time and $x, y,$ and z represent spacial positions. These are **inputs** of a system and should be a set of **independent variables**.

Unless you're modeling the physical movement of people, animals, or goods, you generally won't use vectors to represent space and time. Instead, you'll populate it with data with the independent variables that describe the system you are studying. For example, if you are studying gas milage, you might look at a system like this

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

where x_1 represents the weight of the car, x_2 represents the incline of the road, x_3 represents, x_3 represents the outside temperature, x_4 represents the number of cylinders in the engine, x_5 represents the airpressure of your tires, and x_6 represents the aerodynamics of the car.

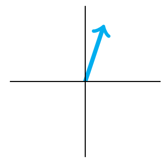
With mechanical systems, like gas mileage of a car, it's not very difficult to determine your variables. You need to know how the system works, which can be figured out by taking it apart.

In the biological, geological and social sciences, it is incredibly difficult to determine your variables! It's not always obvious which variables to use when modeling a system because we don't always know whether a variable is independent of the system or not. For example, suppose we want to model what causes a person to use heroine. We might look at variables like income or the percentage of people in their social network who use heroine. Neither of these, however, is clearly an independent variable. Because heroine use can get in the way of a job, it can influence a person's income. Similarly, heroine users are more likely to hang out with other heroine uses because they may share the risk of purchasing an illegal drug and/or prefer to use the drug in groups.

When a vector doesn't represent movement in space, what does direction mean? It depends. If the variables are different kinds of measurements like a percentage of people in a social network versus an annual income, the direction is meaningless. Direction compares the amounts with one another. If the measurements are not comparable, than direction means nothing. On the other hand, if the two variables are comparing measurements that exist on the same scale, i.e.

(1) measurements all have the same units (like feet) or
(2) values are all percentages,

then the direction allows us to see which variable dominates. For example, in the picture below, the y variable dominates (is larger) over the x variable because the arrow is closer to being parallel to the y-axis.



The magnitude will (in general) not be very helpful for two measurements on the same scale. For measurements which are not on the same scale, there are instances when the magnitude provides important information. For the purposes of this class, however, we will only use magnitude when

(1) discussing spacial vectors, or
(2) when wanting to find a **unit vector**.

7.1. **Unit Vectors.** A unit vector is a vector whose magnitude is 1. For example

$$\vec{v} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

is a unit vector because

$$|\vec{v}| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1.$$

Any vector can be made into a unit vector if we divide by it's magnitude. For example

$$\vec{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

is not a unit vector. It's magnitude is

$$|\vec{w}| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{14}.$$

But if we divide $\vec{w}$ by $\sqrt{14}$, we get a unit vector.

$$\frac{\vec{w}}{\sqrt{14}} = \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \\ \frac{2}{\sqrt{14}} \\ \\ \frac{3}{\sqrt{14}} \end{pmatrix}$$

We can verify that this is a unit vector:

$$\left|\frac{\vec{w}}{\sqrt{14}}\right| = \sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 \left(\frac{3}{\sqrt{14}}\right)^2} = 1$$

We typically try to find a unit vector when we want to only understand direction. This idea will become clearly in later lectures on optimization.

7.2. **Dot Products.** Today we will cover *how* to calculate a dot product, but we won't discuss its meaning until a little later in the course.

A **dot product** is a scalar value that results from taking the sum of the products of respective entries. Given two vectors

$$\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{pmatrix}$$

then their dot product is

$$\vec{v} \cdot \vec{w} = x_1y_1 + x_2y_2 + x_3y_3 + \dots$$

Example 7.1:1. What is the dot product of $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$

Solution 7.2:

$$\vec{u} \cdot \vec{v} = (1)(-1) + (2)(0) + (-3)(-1) = -1 + 3 = \boxed{2}$$

Summary of Ideas: Lecture 7

- Vectors can contain any kind of information. They are typically used to summarize a system.
- In general, magnitude and direction are meaningless.
- A unit vector is any vector of length one.
- Given a any vector $\vec{v}$, we can find a unit vector with the same direction as $\vec{v}$. It is

$$\frac{\vec{v}}{|\vec{v}|}$$

- The **dot product** is a scalar value produced from two vectors. The formula is:

$$\vec{v} \cdot \vec{w} = x_1y_1 + x_2y_2 + x_3y_3 + \dots$$