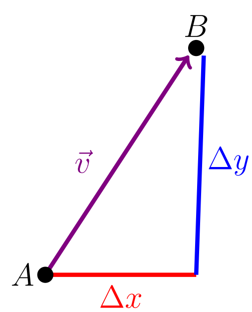
Objectives

- ☐ I understand how to use vectors to understand displacement.
- ☐ I can find the magnitude of a vector.
- ☐ I can sketch a vector.
- ☐ I can add and subtract vector.
- ☐ I can multiply a vector by a scalar.

During the last lecture, we discussed vectors and matrices in the context of systems of linear equations. Now, we will learn a little more about these objects and a different ways of understanding them.

6.1. **Vectors in Physics.** So far, we've learned that vectors are matrices with one column and that each entry is a value for a distinct variable. Because a variable can be any kind of measurement, vectors can represent almost anything.

In physics, vectors are used to indicate a **displacement** of an object from one point A (the initial point) to another point B (the terminal point). Physically, we can represent this as an arrow from A to B. As we saw in the previous section, vectors are typically symbolized by a lower-case letter with an arrow over it like $\vec{v}$. If it represents the displacement from A to B, then we sometimes write it as $\vec{AB}$.



We write this vector as

$$\vec{AB} = \vec{v} = \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

How is this related to the previous lecture? Here, our first variable represents change in position in the x direction and our second variable represents change in position in the y direction. The *net* movement is represented by the vector itself, represented by the violet arrow.

Notice that this is a right triangle, where the arrow is the hypotenuse (the longest leg). The length of $\vec{v}$, denoted $|\vec{v}|$, will tell us the distance from A to B. It can be found using the *pythagorean theorem*[5]:

$$|\vec{v}| = \sqrt{x^2 + y^2}$$

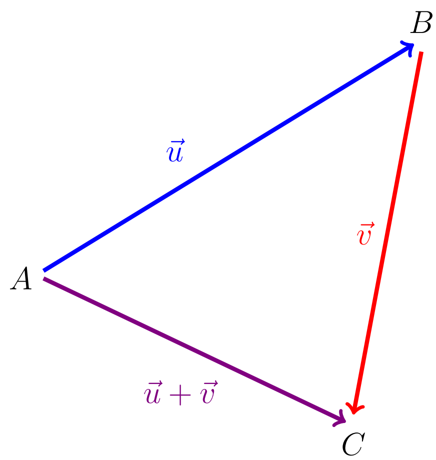
For a generic vector, the "length" of the vector is referred to as **magnitude**.

[5] Recall the identity $a^2 + b^2 = c^2$ from geometry

Check your Understanding

Explain why the zero vector, a vector with only zeros in its entries, should be a point.

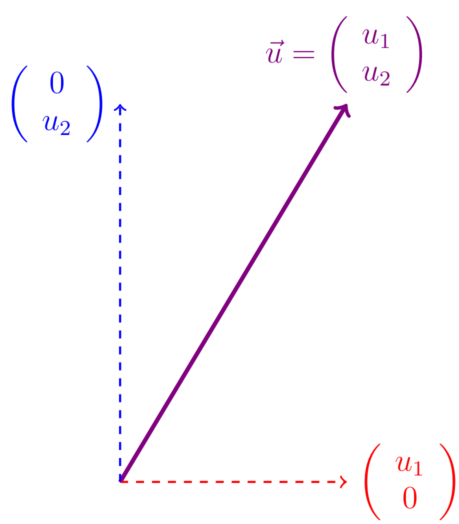
The sum of two vectors can be thought of as a combination of two displacements. For instance, if the vector $\vec{u}$ represents the displacement from point A to point B, and the vector $\vec{v}$ represents the displacement from the point B to point C, then their sum $\vec{u}+\vec{v}$ represents the total displacement from point A to point C. (This is called The Triangle Law.)



We add two vectors component-wise. For $\vec{u}=\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\vec{v}=\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$,

$$\vec{u}+\vec{v}=\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}$$

By this same logic, any vector can be broken up into its horizontal and vertical vectors. That is



Everything works the same way in three dimensions. Below is an example. You can think of the x direction as being forward/backward, the y direction as left/right, and the z direction as up/down.

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6.1.1. *Examples.* **Example 6.1:** A group of scientists are modeling how bats fly in confined spaces. A large 20ft by 20ft by 20ft room is fitted with synchronized cameras hooked to computers which can extrapolate the position of the bat from the images. The middle of the room is the point $(0, 0, 0)$. Two photos taken in succession show the bat at point $A(1, 3, 5)$ and then 10 seconds later at $B(-1, 0, 7)$.

(1) Define the vector from A to B.
(2) Approximate the speed (in feet per second) at which the bat traveled from A to B.
(3) Sketch the room and the vector.

Solution 6.2:

(1) To measure a total distance traveled, we want to subtract our final destination from our starting point.

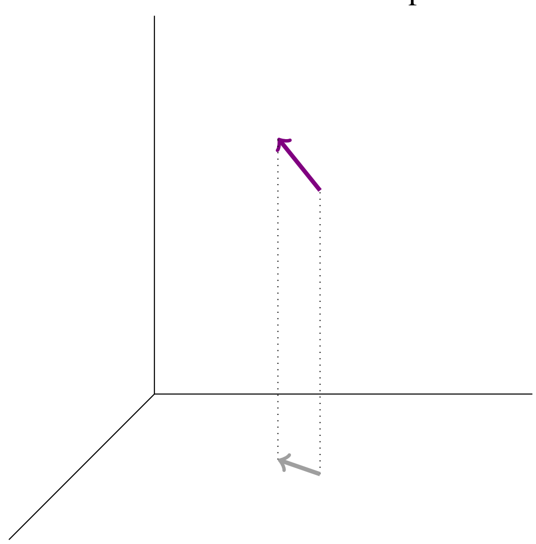
$$\vec{AB} = \begin{pmatrix} -1-1 \\ 0-3 \\ 7-5 \end{pmatrix} = \boxed{\begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}}$$

(2) To determine the speed, we need to know the distance traveled. First, we should remember that all measurements are taken in feet. Next, we need to determine the length of the distance between A and B. This is equal to the *magnitude* of the vector found above. Therefore, we calculate

$$|\vec{AB}| = \sqrt{(-2)^2 + (-3)^2 + (2)^2} = \sqrt{4+9+4} = \sqrt{17} \approx 4 \text{ ft.}$$

The bat traveled approximately 4 feet in 10 seconds or 24 feet per minute.

(3) We want to sketch this movement in 3 dimensions, which isn't easy. It looks roughly like the image below. The shadow and lines are added to help see the depth.



Example 6.3: What is $\vec{v} + \vec{u}$ and $\vec{v} - \vec{u}$ for $\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} -1 \\ 4 \\ -5 \end{pmatrix}$

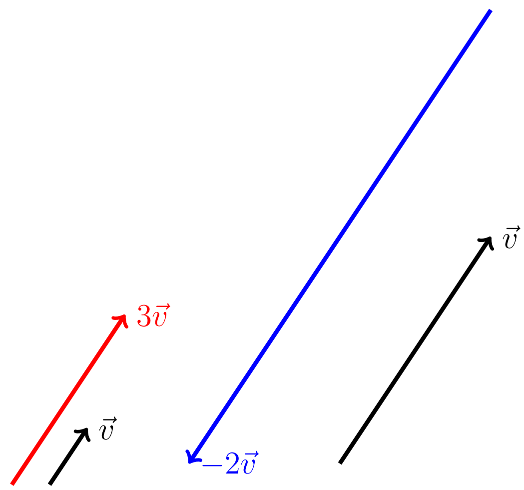
Adding and subtracting vectors works component-wise. Be careful with your signs!

$$\vec{v}+\vec{u} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \\ -5 \end{pmatrix}$$
$$= \begin{pmatrix} 2+(-1) \\ 3+4 \\ 1+(-5) \end{pmatrix}$$
$$= \boxed{\begin{pmatrix} 1 \\ 7 \\ -4 \end{pmatrix}}$$

$$\vec{v}-\vec{u} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \\ -5 \end{pmatrix}$$
$$= \begin{pmatrix} 2-(-1) \\ 3-4 \\ 1-(-5) \end{pmatrix}$$
$$= \boxed{\begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix}}$$

Constants, or what we've called "normal numbers," are called **scalars**. The magnitude of a vector is a scalar. There is also the notion of multiplying a vector by a scalar. For any vector $\vec{v}$ and any scalar c, the scalar multiple $c\vec{v}$ is obtained by multiplying the length of $\vec{v}$ by c and keeping the

- same direction if $c > 0$, or the
- opposite direction if $c < 0$.



Notice that multiplying by a scalar adjusts the magnitude, but doesn't rotate the arrow (it can only flip it).

In the same way we can focus on a **scalar component** of a vector (its magnitude), we can focus on just the direction of a vector. To do this, we consider a vector that points in the same direction but has length 1. This is called a **unit vector**.

6.1.2. *Examples.* **Example 6.4:** Two bottle rockets are launched. The second is four times as fast as the first. They travel in the same direction. The first travels from $(0, 0, 0)$ to $(1, 1, 2)$ in four seconds. Where does the second end up in four seconds if it begins at $(1, 1, 0)$?

To answer this, we begin by defining the vector of displacement for the first rocket. That is

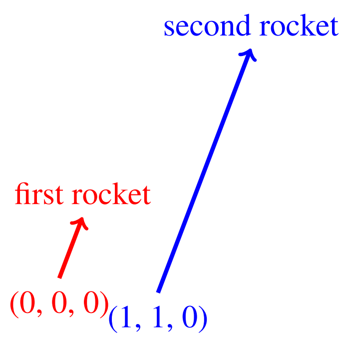
$$\vec{v} = \begin{pmatrix} 1-0 \\ 1-0 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

If the second moves 4 times as fast as the first, then its vector of displacement just multiply each component by the scalar value.

$$4\vec{v} = \begin{pmatrix} 4 \\ 4 \\ 8 \end{pmatrix}$$

This, however, does not tell us the location of the second rocket. It only tells us how it moves in four seconds. If the rocket begins at $(1, 1, 0)$ then it increase by 4 steps in the x direction, 4 steps in the y direction, and 8 steps in the z direction. Therefore, in four seconds the rocket is at

$$(1 + 4, 1 + 4, 0 + 8) = \boxed{(5, 5, 8)}$$



Example 6.5: Find the unit vector in the direction $\vec{v} = \begin{pmatrix} 1, 2, 1 \end{pmatrix}$.

How can we find a unit vector that points in the same direction?
First, let's find the length of $\vec{v}$.

$$|\vec{v}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

If we divide $\vec{v}$ by its magnitude, we get the vector

$$\vec{u} = \left(\frac{1}{\sqrt{6}}\right)\vec{v} = \begin{pmatrix} \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \end{pmatrix}$$

Let's check our work. To do this, we ask ourselves, "What is the magnitude of this vector?"

$$|\vec{u}| = \sqrt{\left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{2}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2} = \sqrt{\frac{1}{6} + \frac{4}{6} + \frac{1}{6}} = 1$$

Notice that $\vec{u}$ is just the vector $\vec{v}$ multiplied by a positive scalar, so the direction doesn't change. So our unit vector is

$$\boxed{\vec{u} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)}$$

Summary of Ideas: Lecture 6

- Vectors can be understood as a description of movement from one point to another.
- The **magnitude** of a vector, denoted $|\vec{v}|$, is its length. We can find it using the pythagorean theorem:

$$|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$$

- Multiplying by a scalar changes the length of a vector. If that scalar is negative, the vector flips its direction.