5. LECTURE 5

Objectives

- ☐ For two or more linear equations, I can find that solution (if one exists).
- ☐ I can determine the meaning of a solution to a system of linear equations.
- ☐ I can rewrite a system of linear equations as a linear expression with vectors and matrices.
- ☐ I can solve matrix equations with the help of my calculator.

Up until now, we have looked at finding linear relationships between two measurements from a set of collected data. Now, we will build the theory we need to understand *how* our calculator finds this line. This technique will be very useful when we look at nonlinear relationships (relationships that are described by something other than a line) as well as related systems where no measurement is clearly dependent like predator and prey populations.

5.1. **Systems of Linear Equations.** A **system of linear equations** is a set of lines in terms of the same variables. Physically, it means we have a set of relationships between the same measurements in different situations. The goal in these problems is always the same: find values of x and y that satisfy *all* the equations in the set.

Example 5.1: A new teacher in State College decides to form an archery club for middle schoolers and high schoolers. In a news article, it explains that middle schoolers begin with a 24 pound recursive bow (this is the draw weight of the shot) while high schoolers begin with a 45 pound recursive bow. The article states that 112 students are involved in the archery club and that Dick's Sporting Goods donated all the bows, which are equal in value to $22,700. You find out online that 24 pound recursive bows are $140 each and 45 pound recursive bows are $270 each. How many middle schoolers and high schoolers are involved in the clubs?

Solution 5.2: The first step to any word problem is to identify the measurements we do not know but wish to find.

There are two unknowns: the number of middle schoolers in the club and the number of high schoolers in the club. We'll use variables x and y for each one, respectively.

What information to we have about these two groups? We know they total 112 because that is the size of the club. Therefore,

$$x + y = 112.$$

The other piece of information we know is that the total number of bows costs $22,700. We know the high schoolers' bows were $270 each while the middle schoolers' bows were $140 each. Therefore,

$$140x + 270y = 22700$$

since $140x$ will equal the *total* amount of money spent on middle schooler bows and $270y$ will equal the *total* amount of high schooler bows.

Notice that we don't have a use for the figures "24 pound" or "45 pound." In this context, these numbers tell us the amount of force needed to use the bows, so they can't help us determine *how many* middle schoolers or high schoolers in the club.

Now that we've exhausted all the numbers listed, we will take our equations and solve for x and y. There are 2 methods to do this:

- **Substitution:** This method requires using one equation to solve for a variable and then plugging that expression into the other equation. Suppose we solve for y in the first equation. Then we get $y = 112 - x$. This expression is substituted into the second equation:

$$\begin{aligned} 140x + 270y &= 22700 \\ 140x + 270(112 - x) &= 22700 \\ 140x + 30240 - 270x &= 22700 \\ -130x + 30240 &= 22700 \end{aligned}$$

  After simplifying, we get the expression $-130x + 30240 = 22700$. When we solve for x in this equation, we get $x = 58$. There are a number of ways I can find y, but the easiest is to use the first equation we wrote: $y = 112 - x$. When we plug in 58 for x, we get $y = 112 - 58 = 54$.

  Hence, there were 58 middle schoolers and 54 high schoolers.

- **Elimination:** This method requires that you combine the equations so as to eliminate one of the variables. Here, we multiply the entire equation $x+y = 112$ by 140 and then subtract the two equations to eliminate x.

$$\begin{array}{rl} & 140(x + y = 112) \\ - & 140x + 270y = 22700 \\ \hline & \end{array} \implies \begin{array}{rl} & 140x + 140y = 15680 \\ - & 140x + 270y = 22700 \\ \hline & -130y = -7020 \end{array}$$

  Now we can solve for y using the equation $-130y = -7020$. When we do, we get $y = 54$. We then plug 54 for y in one of the previous equations. The easiest is $x + 54 + 112$, which gives us $x = 58$.

  Hence, we get the same answer as we do with substitution.

Remark 5.3: There is no significant difference between using elimination and substitution. For problems with many equations, elimination can be faster, but substitution generally proves to be easier for humans. In the second half of these notes, we will learn how to solve these equations with matrices.

Remark 5.4: Notice that when we write equations like $x + y = 112$, we are treating x and y as both independent variables! We technically have a third *dependent* variable z, the population of the entire club, which depends on x and y. We don't explicitly use this variable because we are only concerned with one value: $z = 112$. Later on in the course, we will look at equations like $x + y = z$ and consider many possible solutions.

Example 5.5: Suppose you find an injured Allegheny woodrat, a threatened species in Pennsylvania. You decide to nurse it back to health and then releasing it into the wild. Suppose an ideal

Allegheny woodrat diet contains 12% protein, 4.5% fat, 15% fiber. At the corner store, you have the following options:

- canned dog food: 10% protein, 7.0% fat, 3.0% fiber
- hamster food: 12% protein, 5.0% fat, 18% fiber
- rabbit food: 13 % protein, 3.5% fat, 19% fiber
- (domesticated) rat food: 15% protein, 4.0% fat, 7.0% fiber

What mixture of dog food, hamster food, and rabbit food will be appropriate for the woodrat? Find the amounts for one cup (a week's worth of food).

Solution 5.6:(Partial) First, we need to determine the variables. These are the measurements we can adjust. Since we cannot adjust the percentage of protein in each food, our variables must be the amounts of each food type. Let x be the amount of dog food we will use in the mixture, y the amount of hamster food used, z the amount of rabbit food and w the amount of domesticated rat food used.

The mixture will include dog food (x), hamster food (y), rabbit food (z), and domesticated rat food (w) and needs to add to one cup. Hence

$$x + y + z + w = 1.$$

We now write equations for protein, fat, and fiber.

$$\begin{aligned}
&\text{Protein: } .10x + .12y + .13z + .15w = .12\\
&\text{Fat: } \quad .070x + .050y + .035z + .040w = .045\\
&\text{Fiber: } .030x + .18y + .19z + .070w = .15
\end{aligned}$$

From here, we can determine the values of x, y, z and w using substitution or elimination. As you can see, however, it will get complicated because we have four variables and four equations.

5.2. **Matrix Equations.** Matrices are a way to summarize a system of equations. In the previous example, we have four equations that used the same variables:

$$\begin{aligned}
&x + y + z + w = 1\\
&.10x + .12y + .13z + .15w = .12\\
&.070x + .050y + .035z + .040w = .045\\
&.030x + .18y + .19z + .070w = .15
\end{aligned}$$

A matrix is a kind of table that keeps track of the coefficients in a system of linear equations. The first row corresponds to the coefficients of the first equation and each column corresponds to only one variable. So the above set of equations can be written as

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ .1 & .12 & .13 & .15 \\ .070 & .050 & .035 & .040 \\ .030 & .18 & .19 & .070 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ .12 \\ .045 \\ .15 \end{pmatrix}$$

This arrangement may not seem revolutionary. All we did was:

- take the coefficients and put them in a table, called a **matrix**
- take the variables and put them in a column, called a **vector**, and
- take the solutions from the problem and put them in a column, also a **vector**.

On the surface, it doesn't seem like a big deal. These matrices and vectors, however, can be treated as numbers *and* as functions; this opens up a lot of useful math that makes up this course. For now, we will manipulate these objects using our calculator.

Before we move on to how to techniques with a calculator, we should mention how matrices add and multiply.

5.3. **Facts About Matrices & Vectors.** Before we move on to manipulating matrices with the help of a calculator, let's first talk about some basic facts about these objects. After all, we will be using them later on.

Matrix Size 5.7

> Matrices can come in any size.

The following are all different examples of matrices:

$$\begin{pmatrix} 1 & 2 \\ \pi & 0 \end{pmatrix} \qquad \begin{pmatrix} 4.325 & 97000 & e \\ 0 & 24 & \frac{3\pi}{4} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 & 3 \\ 2 & 0 & \frac{3\pi}{4} & 7 \\ 1 & 1 & 0 & -1 \\ -4 & -4 & 3 & 10 \\ 9 & 0 & -\pi & 5,280 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad (4)$$

We see from the above example that matrices don't have to be squares. They can be longer or taller rectangular tables. The last example shows us that "normal numbers" like 4 are, in fact, just a special kind of matrix.

Vector Size 5.8

> Vectors are columns of numbers with no limit in length. Vectors are a special kind of matrix.

Below are examples of vectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 4.325 \\ \pi \\ 5280 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ 1 \\ -4 \\ 9 \\ 1 \\ 0 \end{pmatrix} \tag{4}$$

We can make two interesting observations about vectors. The first is that any matrix is just a bunch of vectors put together. The second is that “normal numbers,” like 4, are a special kind of vector.

Adding & Subtracting Matrices and Vectors 5.9

> We can only add or subtract matrices/vectors of the same size. When we add/subtract them, we do so entry-wise.

For example, when we add the two matrices below, we are just adding the entries of each corresponding part in the table.

$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 4 & -1 \\ -2 & -4 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ -2 & 12 \\ 10 & 8 \\ 3 & 9 \end{pmatrix} = \begin{pmatrix} 2+1 & 3+4 \\ 1+(-2) & 0+12 \\ 4+10 & -1+8 \\ -2+3 & -4+9 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ -1 & 12 \\ 14 & 7 \\ 1 & 5 \end{pmatrix}$$

Similarly, when we subtract two matrices, we subtract entry-wise.

$$\begin{pmatrix} 2 \\ 4 \\ 3 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -2 \\ -5 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 4-0 \\ 3-(-5) \\ 0-2 \\ -2-(-3) \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 8 \\ -2 \\ 1 \end{pmatrix}$$

Multiplying Matrices 5.10

* To multiply two matrices, we take entries in the row of the first matrix and multiply them by the entries in the column of the second matrix and add up those products.
* For this to work, we need the number of columns of the first matrix to equal the number of rows of the second matrix.

As you can see, multiplying matrices can be a little confusing when you first see them. When in doubt, always go back to our original set-up: a system of linear equations.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ .1 & .12 & .13 & .15 \\ .070 & .050 & .035 & .040 \\ .030 & .18 & .19 & .070 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

When we multiply the entries of the row by the entries of the column and sum them, we end up with our linear equations. When we multiply, we get the sum of the coefficients times the corresponding variable.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0.10 & 0.12 & 0.13 & 0.15 \\ 0.070 & 0.050 & 0.035 & 0.040 \\ 0.030 & 0.18 & 0.19 & 0.070 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x + y + z + w \\ 0.10x + 0.12y + 0.13z + 0.15w \\ 0.070x + 0.050y + 0.035z + 0.040w \\ 0.030x + 0.18y + 0.19z + 0.070w \end{pmatrix}$$

To see this process written out step-by-step, you can visit `https://www.youtube.com/watch?v=aKhhYguY0DQ`[4]

Check your Understanding

Explain why the only vectors you can multiply are just those that are 1 entry (i.e. "normal" number).

A natural next question you might ask is: How do I divide two matrices? In a sense, we don't divide them. We multiply by the inverse matrix (which is just like division). In this course, we will not worry about how to find a matrix's inverse by hand. We will instead do this with a calculator.

5.4. **Matrix Equations on the Calculator.** The goal of this section is find a solution to a system of equations like this.

[4]The speaker says these are "human-created definitions" for matrix multiplication. If you decide to study math further, you'll learn that in some sense, there is no other way matrix multiplication could work. Matrices are a very special way to represent many mathematical objects. So his statement is debatable.

$$
\begin{aligned}
&x + y + z + w = 1 \\
&.10x + .12y + .13z + .15w = .12 \\
&.070x + .050y + .035z + .040w = .045 \\
&.030x + .18y + .19z + .070w = .15
\end{aligned}
$$

That is, we find numerical values for each variable x, y, z and w so that when we plug those numbers into the equations above, we get the solutions 1, .12, .045, and .15 (respectively).

Our technique uses matrix equations. We turn this system of linear equations into the following equation:

$$
\begin{pmatrix} 1 & 1 & 1 & 1 \\ .1 & .12 & .13 & .15 \\ .070 & .050 & .035 & .040 \\ .030 & .18 & .19 & .070 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ .12 \\ .045 \\ .15 \end{pmatrix}
$$

When mathematicians read the above statement, they read it as

$$A\vec{\gamma} = \vec{b}$$

where

$$\vec{\gamma} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 1 \\ .12 \\ .045 \\ .15 \end{pmatrix}.$$

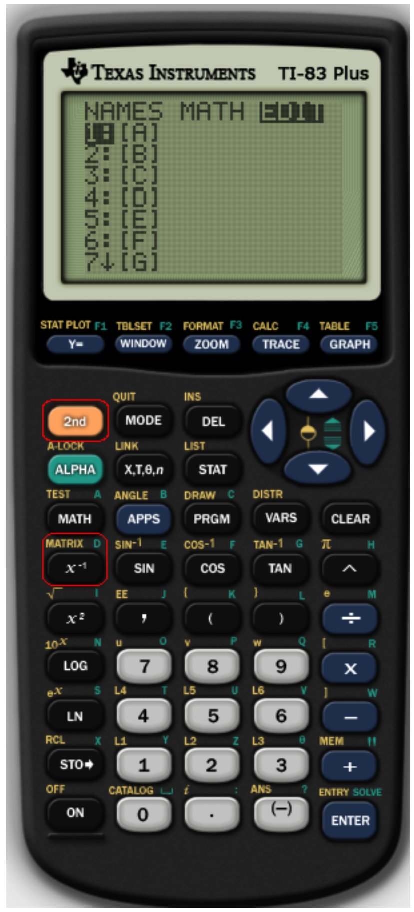
Solving the equation $A\vec{\gamma} = \vec{b}$ amounts to “dividing” both sides by A. For matrices, that’s multiplying by the inverse:

$$A^{-1}A\vec{\gamma} = A^{-1}\vec{b}$$

When we multiply a number (like 2) with its inverse ($\frac{1}{2}$), we always get 1. For matrices, it is the same, except ”1” is a special kind of square matrix with 1s along the diagonal entries and zeros everywhere else. For now, though, we will treat this like we do with any other algebra problem. If it becomes 1, it does not need to be written. So we get the equation:

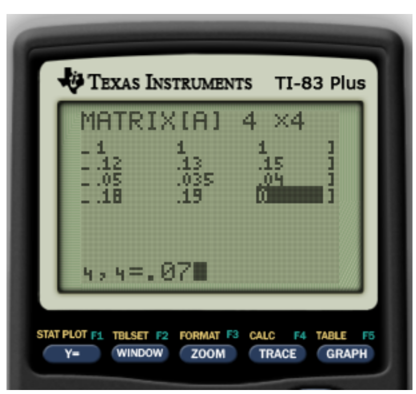
$$\vec{\gamma} = A^{-1}\vec{b}$$

This is what we calculate. So we will find $A^{-1}\vec{b}$ using our calculator. The first step is to enter A and $\vec{b}$ into our calculator. We begin by pressing `2nd` and `x`$^{-1}$ to access the `MATRIX` menu. Navigate at the top using the right arrow to the `EDIT` menu. Make sure `[A]` is highlighted and hit enter.



When you enter the matrix menu for [A], you'll need to first enter the dimensions of the matrix. The first number is the number of rows. The second is the number of columns. For A, we need 4 rows and 4 columns. Enter the values that correspond to the same location:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ .1 & .12 & .13 & .15 \\ .070 & .050 & .035 & .040 \\ .030 & .18 & .19 & .070 \end{pmatrix}$$



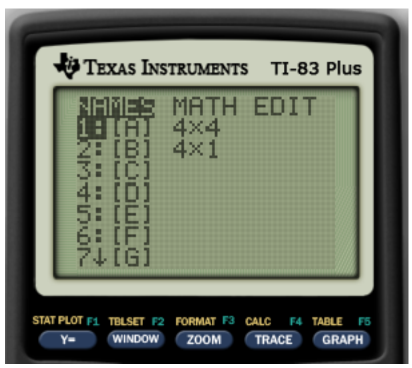
Once you're done, press 2nd and MODE to quit. Repeat the process to enter the values in $\vec{b}$ by returning to the MATRIX menu, navigating to EDIT and selecting [B]. This will have 4 rows and 1 column.

Once you're done, press 2nd and MODE to quit. The calculator now knows the matrix A and the vector $\vec{b}$. What is left is to compute

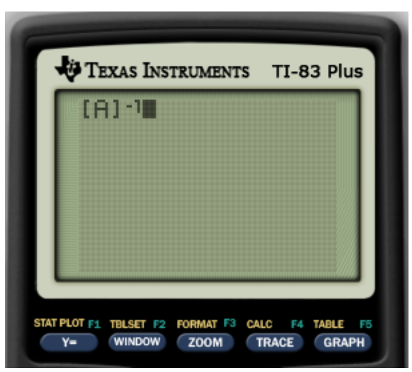
$$A^{-1}\vec{b}$$

to find the solution to the system of linear equations.

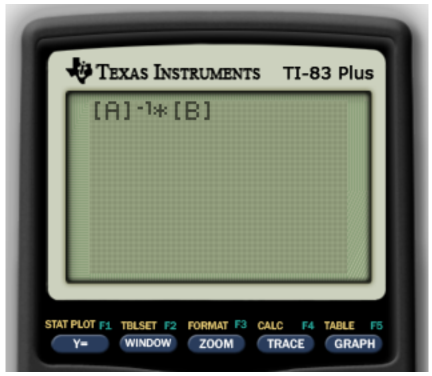
To do this, we will return to the MATRIX menu by pressing 2nd and x^{-1}. This time, we stay in the NAMES menu and select 1: [A] 4 x 4.



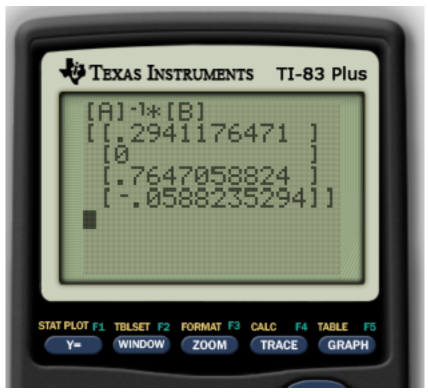
Then [A] will appear on the home screen. Press x^{-1} to display $[A]^{-1}$.



Press the button for multiplication and return to the MATRIX menu, selecting [B] under NAMES. Once you do, your home screen should look like the image below.



When you press enter, you'll see a display of a vector for the values of $\vec{\gamma}$. Here is what we get:



That means

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0.29 \\ 0 \\ 0.74 \\ -0.05 \end{pmatrix}$$

Going back to the original problem, our solution suggest that the ideal mixture for the rat is .29 cups of dog food, 0 cups of hamster food, .74 cups of rabbit food, and -.05 cups domesticated rat food. This last figure, as we know, is not possible. There is no way we can subtract rat food from a mixture of dog food and rabbit food.

So what went wrong? We found a solution but it isn't realistic. That's because no where in our technique did we include the restriction that x, y, z, and w be positive. Once we do, we're looking at an *optimization* problem. That's where we want to find the "best" solution with some set of physical limitations. The main point of this class will be tackling these kinds of problems.

For now, however, we will look at simple systems of linear equations. Sometimes our solutions will make physical sense, but sometimes they won't.

Summary of Ideas: Lecture 5

- A **system of linear equations** is a set of lines such that the variables used in all the equations represent the same physical measurements.
- There are two methods of solving systems of linear equations by hand: **substitution** and **elimination**.
- Numerically, we solve these problems using matrices.
- A **matrix** is a table of numbers that can be manipulated like a "normal number." A **vector** is a matrix with only one column.
- Two matrices can be added or subtracted only if they have the same dimensions (same number of rows and columns). We add matrices entry-wise.
- Two matrices can be multiplied if the number of columns of the first matrix equals the number of rows of the second one.
- To solve a system of linear equations using matrices, we define a matrix based on the coefficients, A, a vector with all the variables $\vec{\gamma}$, and a vector with the solutions to the linear equations $\vec{b}$. Then we calculate

$$\vec{\gamma} = A^{-1}\vec{b}$$