Objectives

- ☐ I know that predator-prey systems model situations where the flows depend on the stocks of two populations.
- ☐ I can solve for the non-trivial solution of the system.
- ☐ I can show that a solution is stable.

Last week, we discussed cooperative games. These were characterized by two players hoping to get an optimum amount of the same thing–a higher grade, a lower prison sentence, or more milk. Because each player's choices are tied with the other's, both players can paradoxically end up worse off when they both try to maximize.

Now, we consider the predator-prey model. It's used to model two groups which are truly competing at the expense of the other.

What happens in a predator-prey relationship? Suppose you have an island where the only animals are foxes and rabbits. If the system is in balance, the population will fluctuate in size. In particular,

- when there are a lot of rabbits, the foxes have an easier time to eat them; more of the fox population survives and reproduces. (foxes: ↑, rabbits: ↓)
- when there are many foxes and few rabbits, more foxes starve; as the fox population shrinks, the rabbit population recovers. (foxes: ↓, rabbits: ↑)

The best way to model this process is using differential equations. These are equations that tell us how the derivatives change. Let V be the population of rabbits (victims) and P be the population of foxes (predators). Then

$$\frac{dV}{dt} = bV - aVP$$

where b is the "birthrates" of rabbits and n is the rate at which foxes consume rabbits. What this equation says is that the population of rabbits increases with the birth rate and decreases with the rate at which rabbits and foxes interact.

Similarly,

$$\frac{dP}{dt} = (ca)VP - dP$$

where d is the "death rate" of foxes, a is the rate at which foxes consume rabbits (i.e. the amount they are eaten), and c reflects the number of eaten prey needed to sustain a fox.

We write "birthrate" and "death rate" in quotations because that's not entirely accurate. The constant b really tells us how the rabbit population would grow if the foxes magically disappeared. The constant d tells us the rate at which the foxes would die if the rabbits magically disappeared.

Therefore, the entire model can be summarized by *two* equations of the derivative with 4 constants:

$$\begin{cases} V'(t) = bV - aVP \\ P'(t) = (ca)VP - dP \end{cases}$$

These two equations tell us that the *flows* for the population of rabbits and foxes depends on the *stocks* of each population.

Remark 29.1: Although we are thinking about this in terms of literal predators and literal prey, we can extend this concept to other situations where the flows depend on the stocks and two (or more) groups compete. In one of the examples, we will see this applied to employment.

Why model predator-prey relationships with such a detailed model? In a broad sense, we know how the relationship can go; however, these two populations may be gradually headed for collapse! If as know the constants of the system, we find out if the populations are **stable**, meaning they will continue to persist so long as nothing changes, or **unstable**, meaning that under the current conditions, the predators or both the predators and prey will die off.

To classify whether they are stable or unstable, we need to do a rather complicated process. Here are the steps;

(1) Factor $V'(t)$ and $P'(t)$. Set both equations equal to zero (stability points are just critical points!).
(2) Solve for V and P in the above equations so that neither is equal to zero.
(3) Define the Jacobian matrix

$$J = \begin{bmatrix} V'_V & V'_P \\ P'_V & P'_P \end{bmatrix}$$

(4) Plug in the point from (2) into the Jacobian matrix in (3).
(5) Find the eigenvalues of (4).
(6) Determine stability based on those eigenvalues. Suppose your eigenvalues are $a \pm bi$.
   - If $a = 0$, then the system is stable,
   - If $a \neq 0$, it is unstable.

Let's explore this process further by considering some examples.

Example 29.2: Suppose Google and Yahoo compete for workers. Although workers can retire, let's say that Yahoo recruits 1 new worker for every worker it has. For every worker Google has, they have a net loss of one worker from retirement. Google actively tries to recruit workers from Yahoo, with an interaction rate of 2. What is the predator-prey model that describes this relationship?

Solution 29.3: Here, the predator is Google and the prey is Yahoo. Based on the numbers given, we have the set of equations

$$\begin{cases} V'(t) = V - 2VP \\ P'(t) = 2VP - 1P \end{cases}$$

Our first step is to factor these equations and set them equal to zero.

$$\begin{cases} V'(t) = V(1 - 2P) = 0 \\ P'(t) = P(2V - 1) = 0 \end{cases}$$

Next, we want to solve these equations. We can easily see that $V = 0$ and $P = 0$ is a solution. But this solution says the population of Google and Yahoo is zero! We aren't interested in that stable point. Instead, we are interested in the one where both populations are nonzero. So the solution we're interested in is $P = 1/2$ and $V = 1/2$ (verify that plugging in these numbers yields zero for the above equations).

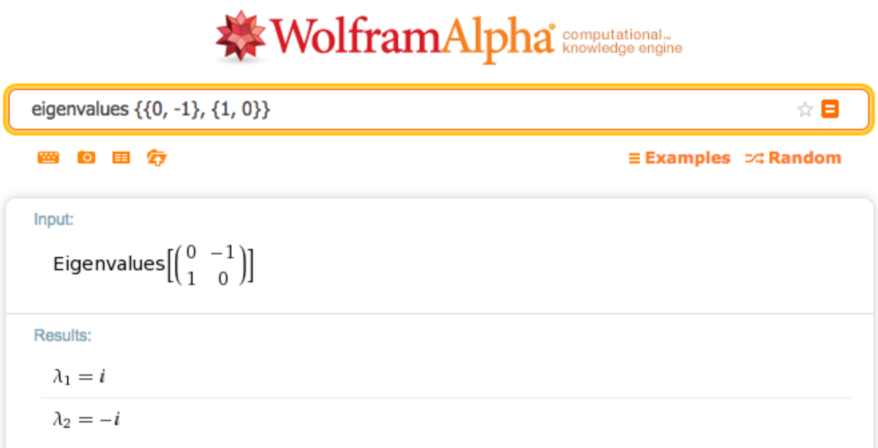
We then define the Jacobian matrix.

$$J = \begin{bmatrix} 1-2P & -2V \\ 2P & 2V-1 \end{bmatrix}$$

Now, we plug in our point $P = 1/2$ and $V = 1/2$ into J.

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Finally, we calculate the eigenvalues. This can be done with Wolfram Alpha.



This can also be done by hand. Let x be your eigenvalue. Then

$$0 = \det \begin{bmatrix} 0-x & -1 \\ 1 & 0-x \end{bmatrix} \implies (-x)(-x) - (-1)(1) = 0 \implies x^2 + 1 = 0$$

When we solve for $x^2 + 1 = 0$, we get $x = \pm\sqrt{-1} = \pm i$.

Therefore, the system is stable when half of the employees are at Yahoo and half are at Google.

Example 29.4: Determine if the system is stable or unstable.

$$\begin{cases} V'(t) = 3V - 5VP \\ P'(t) = 10VP - 4P \end{cases}$$

Solution 29.5:

$$\begin{cases} V'(t) = V(3-5P) = 0 \\ P'(t) = 2P(5V-2) = 0 \end{cases}$$

When we solve, we get $P = 3/5$ and $V = 2/5$).

The Jacobian matrix is

$$J = \begin{bmatrix} 3-5P & -5V \\ 10P & 10V-4 \end{bmatrix}$$

When we plug in, we get

$$J = \begin{bmatrix} 0 & -1 \\ 6 & 0 \end{bmatrix}$$

The eigenvalues are

$$0 = \det \begin{bmatrix} 0 - x & -1 \\ 6 & 0 - x \end{bmatrix}$$

which means $(-x)^2 - (-1)(6) = 0$ or

$$x^2 + 6 = 0$$

.

The eigenvalues are, therefore, $x = \pm\sqrt{-6} = \pm i\sqrt{6}$.

Since the real part is zero, we know that the system is stable when 3/5 of the population are predators and 2/5 of the population are prey.

Summary of Ideas: Lecture 29

- Predator-prey models are systems of equations that describe how the flows of two populations are related to each population's stock.
- Each system will have a nonzero solution that reflects the proportion of each species.
- We can prove that this solution is stable by computing the Jacobian matrix, plugging in the point, and finding the eigenvalues. When the eigenvalues have a real part of zero, the system is stable. If the real part is nonzero, the system is unstable.