

## 26. LECTURE 26

### Objectives

- I understand the idea behind the Method of Lagrange Multipliers.
- I can use the method of Lagrange Multipliers to maximize a multivariate function subject to a constraint.

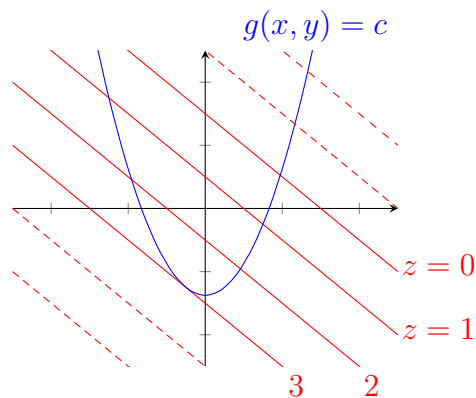
Suppose you have a function  $f$ . Back in Lecture 22, we discussed how to optimize a function like  $f$  over a closed region. This required that we found a critical point inside the region and then look at the boundary of that region.

When we wanted to find an extreme point on the boundary, we had to define each line on the boundary, plug in that line into our function, and find the extreme points. It was a long process.

Today, we learn a different method for finding the critical points of a function on some boundary. This method is called **Lagrange Multipliers**. It can sometimes be easier to use than the "plugging in" method we saw before, especially when

- you are considering more than two independent variables, and
- the boundary can be described by just one function,  $g$ .

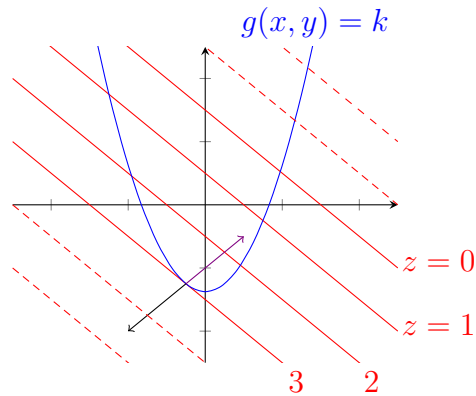
The concept is a little technical, but is easy to see in a picture. Below, we have a function  $f(x, y) = z$  whose level curves are in red. We also have a constraint  $g(x, y) = k$  in blue. Notice that  $f$  equals a variable  $z$ , which  $g$  equals a constant  $k$ . Restricted to the constraint, what is the largest value  $f$  achieves on  $g$ ? According to the picture, the largest value is 3.



At that point, something remarkable happens. The gradients of both curves are parallel! That means, an extreme point happens when

$$\nabla f = \lambda \nabla g.$$

A fair question to ask is, where did  $\lambda$  come from? Notice that at the maximum point in the picture, we can only say the gradients are parallel. That means they have the same direction but possibly not the same magnitude! For example, the picture below provides gradients that are parallel but have different magnitudes. Including  $\lambda$ , which may be negative, helps account for this.



The following steps constitutes the method of Lagrange multipliers:

- (1) Find  $\nabla f$  and  $\nabla g$  in terms of  $x$  and  $y$ .
- (2) Set up the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$g(x, y) = k$$

This will give you a system of equations based on the components of the gradients.

- (3) Solve this system of equations to get  $x$ ,  $y$ , and possibly  $\lambda$ . (There may be multiple solutions.)
- (4) For each of these solutions, find  $f(x, y)$  and compare the values you get. Every solution that gives a maximum value is a maximum point, and every solution that gives a minimum value gives a minimum point.

**Example 26.1:** Three alleles (alternative versions of a gene) A, B, and O determine the four blood types A (AA or AO), B (BB or BO), O (OO), and AB. The Hardy-Weinberg Law states that the percentage of the population who carry two different alleles is

$$P = 2pq + 2pr + 2rq$$

where  $p$ ,  $q$ , and  $r$  are the proportions of A, B, and O in the population. Use the fact that  $p+q+r = 1$  to show that  $P$  is at most  $\frac{2}{3}$ .

**Solution 26.2:** Notice that we have a function of three independent variables

$$P(p, q, r) = 2pq + 2pr + 2rq$$

and a constraint

$$p + q + r = 1.$$

The method we discussed in Lecture 22 would require that we try to substitute and then find critical points. As you might guess, this would be an exhausting calculation. A better approach is to use the Method of Lagrange Multipliers. Let's begin by finding  $\nabla f$  and  $\nabla g$ .

$$\nabla f = \begin{pmatrix} 2q + 2r \\ 2p + 2r \\ 2p + 2q \end{pmatrix}$$

$$\nabla g = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Now, if

$$\nabla f = \lambda \nabla g$$

then we get the following equations:

- $2q + 2r = \lambda$
- $2p + 2r = \lambda$
- $2p + 2q = \lambda$
- $p + q + r = 1$

We want to solve for  $p$ ,  $q$ , and  $r$  using these equations. If it benefits us, we can also solve for  $\lambda$ .

If we take the first two equations, we can combine them:

$$2q + 2r = 2p + 2r.$$

Once we simplify by canceling out  $2r$  and dividing by 2, we get

$$q = p.$$

If we take the second and third equation, we get

$$2p + 2r = 2p + 2q$$

Once we simplify by canceling out  $2p$  and dividing by 2, we get

$$r = q.$$

From these to calculations, we get

$$p = q = r.$$

If we substitute  $r$  for  $p$  and  $q$  in  $p + q + r = 1$ , we get  $3r = 1$ . Therefore  $p = q = r = \frac{1}{3}$ . So the extreme point is  $(1/3, 1/3, 1/3)$ .

According to the problem, we want a maximum. How can we know if this is a maximum? We do this by testing a point that *satisfies* the constraint.

Our point:

$$f(1/3, 1/3, 1/3) = \frac{2}{9} + \frac{2}{9} + \frac{2}{9} = \frac{6}{9} = \frac{2}{3}$$

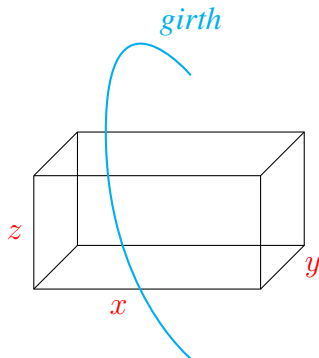
Another (random, made-up) point on  $g$  is  $(1, 0, 0)$ . Notice that  $g(1, 0, 0) = 1$ . What's the proportion according to the Hardy-Weinberg Law?

$$f(1, 0, 0) = 0 + 0 + 0 = 0$$

Since our extreme point  $(1/3, 1/3, 1/3)$  yields a bigger value than our random point  $(1, 0, 0)$ , we know that the point we found was a **maximum**!

**Example 26.3:** A package in the shape of a rectangular box can be mailed by the US Postal Service if the sum of its length and girth (the perimeter of a cross-section perpendicular to the length) is at most 108 in. Find the dimensions of the package with largest volume that can be mailed.

**Solution 26.4:** Let's begin with a picture.



The girth of the package is  $2y + 2z$ . The length is  $x$ . So we have the constraint

$$x + 2y + 2z = 108$$

and we'd like to maximize volume. That is, we want to maximize

$$V(x, y, z) = xyz$$

Let's begin by finding the gradients:

$$\nabla V = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$$
$$\nabla g = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

From  $\nabla V = \nabla g$ , we get

- (1)  $yz = \lambda$
- (2)  $xz = 2\lambda$
- (3)  $xy = 2\lambda$
- (4)  $x + 2y + 2z = 108$

From the second equation, we can solve for  $x$  and find

$$x = \frac{2\lambda}{z}$$

We can substitute this into the third equation to get

$$\frac{2\lambda y}{z} = 2\lambda$$

which can be simplified to  $y = z$ . If we substitute this in equation 3, we get

$$y^2 = \lambda$$

which can be simplified to  $y = \pm\sqrt{\lambda}$ . Because  $y$  is a measurement, it must be  $y = \sqrt{\lambda}$ .

If  $y = \sqrt{\lambda}$ , then  $z = \sqrt{\lambda}$  and  $x = 2\sqrt{\lambda}$ .

Now, we can substitute this into the fourth equation to get

$$2\sqrt{\lambda} + 2\sqrt{\lambda} + 2\sqrt{\lambda} = 108$$

If we simplify, we get

$$\sqrt{\lambda} = 18.$$

Therefore,  $x = 36$ ,  $y = 18$ , and  $z = 18$ . The volume is therefore

$$V(36, 18, 18) = (36)(18)^2 = 11,664.$$

That is, the maximum volume is 11,664 cubic inches, or approximately 6.74 cubic feet.

#### Summary of Ideas

- Lagrange multipliers allow you to maximize a function  $f(x, y)$  subject to a constraint  $g(x, y) = k$ .
- It is a method that can be used to find the extreme points of a function on the boundary of a closed region.
- The method asks you to solve for  $x$ ,  $y$ , and  $\lambda$  given the following expressions

$$\nabla f = \lambda \nabla g \text{ and } g(x, y) = k$$

- We plug in to determine what points are maximums and what points are minimums.