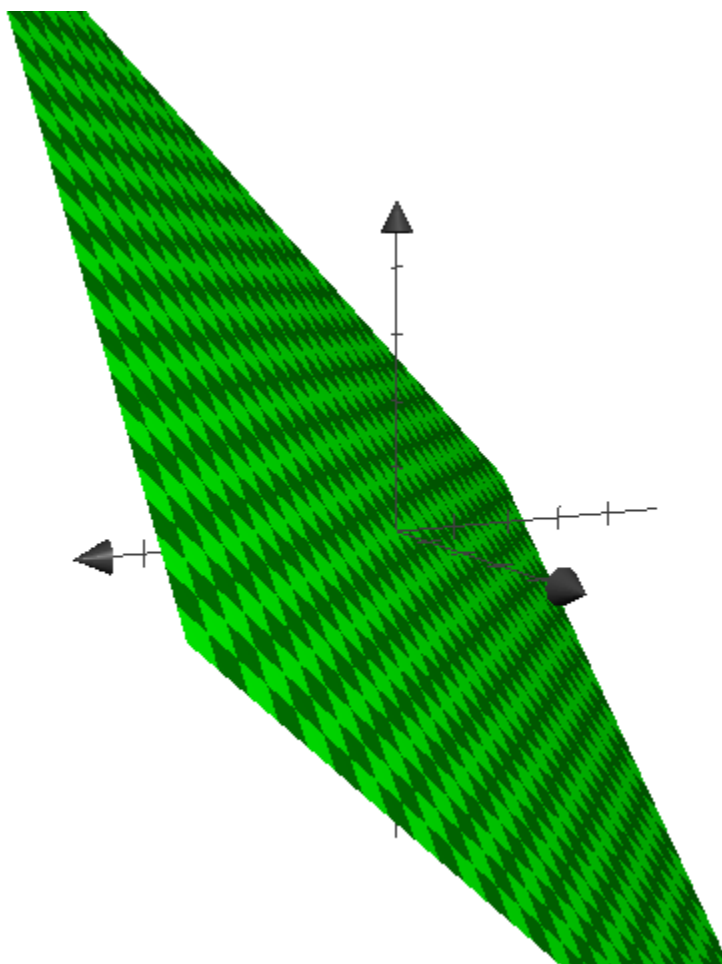


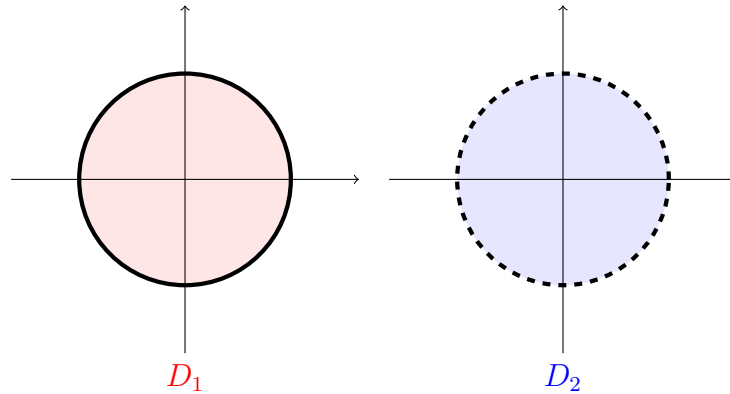
Objectives

- I know what a closed region is.
- I can classify critical points over a closed domain.
- I can determine if a critical point is an absolute maximum or absolute minimum.

In the last lecture, we defined absolute maximum and an absolute minimum. These were critical points that were the highest and lowest values (resp.) a function takes on. These are often positive and negative infinity. For example, for an infinite plane has an absolute maximum of ∞ and an absolute minimum of $-\infty$.



When will the absolute maximum or absolute minimum be finite? If the **domain** (the set of independent variables) is **closed**, then f has a finite **absolute minimum** and **absolute maximum**. A **closed domain** is a set of independent variables that includes its boundary points. For example $D_1 = \{(x, y) | x^2 + y^2 \leq 2\}$ is closed because it includes its boundary while $D_2 = \{(x, y) | x^2 + y^2 < 2\}$ is not closed because it does not.



To find the absolute maximum and absolute minimum, follow these steps:

- (1) Find the the critical points of f on D .
- (2) Find the extreme values of f on the boundary of D .
- (3) The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

The most challenging part of these problems will be considering the values of f on the boundary. You will reduce the problem in one of two ways. Let's consider the domain D_1 above and try to maximize $f(x, y) = xy$ on its boundary.

- One method is to solve one variable in terms of another. The boundary is $2 = x^2 + y^2$, so we could solve and say $y = \pm\sqrt{2 - x^2}$. Then we can plug in for y to get $f(x, y) = f(x) = \pm x\sqrt{2 - x^2}$. The boundary's critical points are precisely those values of x for which

$$0 = f'(x) = \mp \frac{2(x^2 - 1)}{\sqrt{2 - x^2}}$$

This is only true when $x = \pm 1$. We then find the corresponding values of y and find the extreme points on the boundary are

$$(1, 1), (1, -1), (-1, -1), \text{ and } (-1, 1)$$

- Alternatively, we could parameterize the boundary. That means we pick $x = \sqrt{2} \sin t$ and $y = \sqrt{2} \cos t$. Then we get $f(t) = x(t)y(t) = 2 \sin t \cos t = \sin 2t$. We find the critical points of $f(t)$ by solving $0 = f'(t) = 2 \cos 2t$. Here, we need to consider all values of t between 0 and 2π because that is a full rotation around the boundary. Therefore, this is true if $t = \pi/4, 3\pi/4, 5\pi/4, \text{ and } 7\pi/4$. That means our extreme points are $(\sqrt{2} \sin t, \sqrt{2} \cos t)$ for those values of t . That is,

$$(1, 1), (1, -1), (-1, -1), \text{ and } (-1, 1)$$

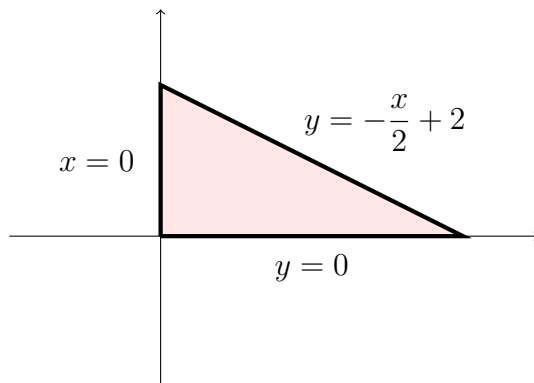
Either approach will work. I recommend you use the one that makes the most sense to you in the given problem.

Before we look at examples, let's briefly discuss terminology. When we say "critical points," we mean points where the derivative or gradient equals zero ($f'(x) = 0$ or $\nabla f = \vec{0}$). We use the term extreme value to just mean the biggest or smallest. The distinction is that an extreme value may not make the derivative zero, but it still may give the largest value.

Example 23.1: Find the absolute maximum and minimum values of the function on D , where D is the enclosed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$.

$$f(x, y) = x + y - xy$$

Let's first draw a picture of D to help us visualize everything.



Solution 23.2: First, we find the critical points on D . We begin by finding the partials and setting them equal to zero

- $f_x(x, y) = 1 - y = 0$
- $f_y(x, y) = 1 - x = 0$

The only critical point on D is $(1, 1)$. Notice that $f(1, 1) = 1$

Next, we find the extreme points on the boundary. We will use the information in our picture to help us.

From $(0, 0)$ to $(0, 2)$, the line is $x = 0$. We can then plug in $(0, y)$, where $0 \leq y \leq 2$. When we plug in these values, we see that

$$f(0, y) = 0 + y - (0)(y) = y.$$

The maximum value this can be is 2, which is achieved at $(0, 2)$. The minimum value is 0, which is achieved at $(0, 0)$.

From $(0, 0)$ to $(4, 0)$, the line is $y = 0$. We can then plug in $(x, 0)$, where $0 \leq x \leq 4$. Along this line, the values are

$$f(x, 0) = x$$

The maximum value is 4, which is achieved at $(4, 0)$. The minimum value is 0, which is achieved at $(0, 0)$.

From $(4, 0)$ to $(0, 2)$, the line that defines it is $y = -x/2 + 2$ for $0 \leq x \leq 4$. Let's plug in.

$$f(x) = x + \left(-\frac{x}{2} + 2\right) - x\left(-\frac{x}{2} + 2\right) = \frac{x^2}{2} - \frac{3x}{2} + 2$$

The critical points are the values of x such that

$$0 = f'(x) = x - \frac{3}{2}$$

The critical point is then $(3/2, 5/4)$ and $f(3/2, 5/4) = -1$. We do not need to check the end points since we already know those values.

Now, let's take a moment to study all the critical points we've found:

- $f(1, 1) = 1$
- $f(0, 2) = 2$
- $f(0, 0) = 0$
- $f(4, 0) = 4$
- $f(3/2, 5/4) = -1$

Therefore, the absolute maximum happens at $(4, 0)$ and the absolute minimum happens at

$$(3/2, 5/4).$$

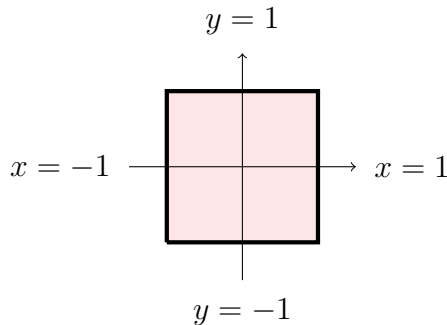
Example 23.3: Find the absolute maximum and minimum values of f on the set D , where

$$f(x, y) = x^2 + y^2 + x^2y + 4$$

and

$$D = \{(x, y) \mid |x| \leq 1, |y| \leq 1\}$$

Solution 23.4: We start with the same process as before. While it's not required, it's always good to start with a picture of your domain.



Let us first find the critical points in D . We solve the following equations

- $f_x(x, y) = 2x + 2xy = 2x(1 + y) = 0$
- $f_y(x, y) = 2y + x^2 = 0$

The critical points are therefore $(0, 0)$ and $(\sqrt{2}, -1)$. We should take a moment to observe that $f(0, 0) = 4$ and $f(\sqrt{2}, -1) = 5$.

Now, let's consider the extreme points of the boundary.

Let's begin with the section $x = -1$. Plugging in gives us the function

$$f(-1, y) = 1 + y^2 + y + 4 = y^2 + y + 5$$

where $-1 \leq y \leq 1$. What is the maximum and minimum value on this side of the box? We can find this by looking for its critical points!

This function achieves its critical point when

$$f'(y) = 2y + 1 = 0 \implies y = -1/2$$

so at $(-1, -1/2)$ is a critical point on the boundary. Let's take note that $f(-1, -1/2) = 19/4$ or 4.75.

When we use derivatives to find critical points, we must also check the end points. Why? Because this is the same process. We're finding the critical points over a close set (a line segment). In this case, we get $(-1, -1)$ and $(-1, 1)$ as potential critical points. Note, their values are $f(-1, -1) = 5$ and $f(-1, 1) = 7$.

Now let's check $x = 1$. Then,

$$f(y) = 1 + y^2 + y + 4 = y^2 + y + 5$$

By the exact same work above, we know $(1, -1/2)$ will be a critical point for the boundary. We, again, take note that $f(1, -1/2) = 19/4$. We also need to check the corners because these are extreme points: $f(1, -1) = 5$ and $f(1, 1) = 7$.

At this point, we've looked at *all* the corners. For the two remaining boundary lines, we can skip this step.

Let us now check $y = 1$. This gives us

$$f(x) = x^2 + 1 + x^2 + 4 = 2x^2 + 5$$

If we consider its derivative, $f'(x) = 4x$, we see that we have a critical point at $(4, 1)$. We should check that $f(0, 1) = 4$.

Finally, we check $y = -1$. This gives us

$$f(x) = x^2 + 1 - x^2 + 4 = 5$$

This is a flat line, so it has no critical points!

Now, let's tally all the points we found.

Critical points of the Surface in D

- $f(0, 0) = 4$
- $f(\sqrt{2}, -1) = 5$

Critical points on the boundary

- (On $x = -1$): $f(-1, -1/2) = 19/4 = 4.75$
- (On $x = 1$): $f(1, -1/2) = 19/4$
- (On $y = 1$): $f(0, 1) = 4$

Corner Values

- $f(-1, -1) = 5$
- $f(-1, 1) = 7$
- $f(1, -1) = 5$
- $f(1, 1) = 7$

Therefore, the absolute maximum is achieved at two locations: $(-1, 1)$ and $(1, 1)$ and the absolute minimum is also at two locations: $(-1, -1/2)$ and $(1, -1/2)$.

Finding absolute maximums and absolute minimums can be quite challenging. As the domains become more complex, so do the calculations. Let's look at one such example.

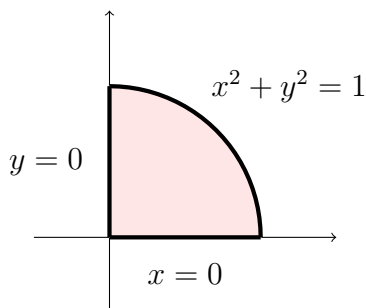
Example 23.5: Find the absolute maximum and minimum values of f on the set D , where

$$f(x, y) = xy^3$$

and

$$D = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$$

Solution 23.6: This domain will be just a quarter of a circle since we restrict both x and y to be greater than zero.



Let us first find the critical points of the function.

- $f_x(x, y) = y^3 = 0$
- $f_y(x, y) = 3xy^2 = 0$

Notice that $y = 0$ is a sufficient condition to get a critical point. That is, all points $(x, 0)$ are critical! To stay in our domain, we'll consider $0 \leq x \leq \sqrt{1 - y^2} = 1$.

$$f(x, 0) = 0$$

Now, let's consider the boundary. You may have noticed that we already considered $y = 0$ above. Let's look at $x = 0$.

$$f(0, y) = 0$$

This doesn't have critical points, and all values are equal to zero (like on $y = 0$). While this may be a frustrating result, we know this problem must have a maximum and minimum. Let's check the last portion of the boundary.

The easiest way to find the critical point on the arc is to parametrize. In particular, $x = \cos t$, $y = \sin t$, and let $0 \leq t \leq \pi/2$.

Since this is easier, let's leave that for you to try on your own. Here, we will use the alternative approach $x = \sqrt{1 - y^2}$ where $0 \leq y \leq 1$.

Let's find the critical points by plugging this equation into $f(x, y)$.

$$\begin{aligned} f(\sqrt{1 - y^2}, y) &= y^3 \sqrt{1 - y^2} \\ \implies f'(\sqrt{1 - y^2}, y) &= 3y^2 \cdot \sqrt{1 - y^2} + y^3 \cdot \frac{1}{2}(1 - y^2)^{-1/2} \cdot -2y \\ &= 3y^2 \sqrt{1 - y^2} + \frac{y^3 \cdot -y}{\sqrt{1 - y^2}} \\ &= 3y^2 \sqrt{1 - y^2} \cdot \frac{\sqrt{1 - y^2}}{\sqrt{1 - y^2}} + \frac{-y^4}{\sqrt{1 - y^2}} \end{aligned}$$

$$f'(\sqrt{1-y^2}, y) = \frac{3y^2(1-y^2) - y^4}{\sqrt{1-y^2}} = 0$$

The above equation is zero when the numerator is zero. That means

$$0 = 3y^2(1-y^2) - y^4 = 3y^2 - 3y^4 - y^4 = 3y^2 - 4y^4 = y^2(3-4y^2)$$

There are two possibilities: $y = 0$ or $y = \frac{\sqrt{3}}{2}$. By plugging into $x = \sqrt{1-y^2}$, we can get two

critical points, $(1, 0)$ and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Now, let's list our critical points and their corresponding values.

- $f(x, 0) = 0$
- $f(1, 0) = 0$
- $f\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \left(\frac{1}{2}\right) \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{4} \approx 1.3$

Hence, the absolute minimum 0 occurs at $(x, 0)$ and $(1, 0)$. The absolute maximum ≈ 1.3

occurs at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Summary of Ideas

- Over a closed region D , you can find the absolute minimum and absolute maximum. These are the smallest and largest values achieved by $f(x, y)$, respectively.
- To find these values, we first find the critical points on D . We then restrict f to the boundary of D and find the extreme values. We can solve one variable in terms of another and plug in the expression or we can parametrize the path and plug in $(x(t), y(t))$.