

22. LECTURE 22

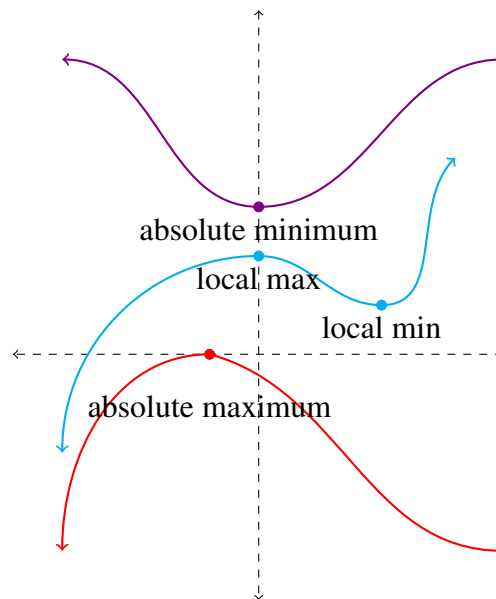
Objectives

- I can define critical points.
- I know the difference between local and absolute minimums/maximums.
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In many physical problems, we're interested in finding the values (x, y) that maximize or minimize $f(x, y)$.

Recall from your first course in calculus that **critical points** are values, x , at which the function's derivative is zero, $f'(x) = 0$. These x -values either maximized $f(x)$ (a **maximum**) or minimized $f(x)$ (a **minimum**).

We did not simply call a critical point a maximum or a minimum, however. Sometimes your critical point is local, meaning it may not be the highest/lowest value achieved by the *entire* function, but it's the highest/lowest point "near by." See the image below for clarification.



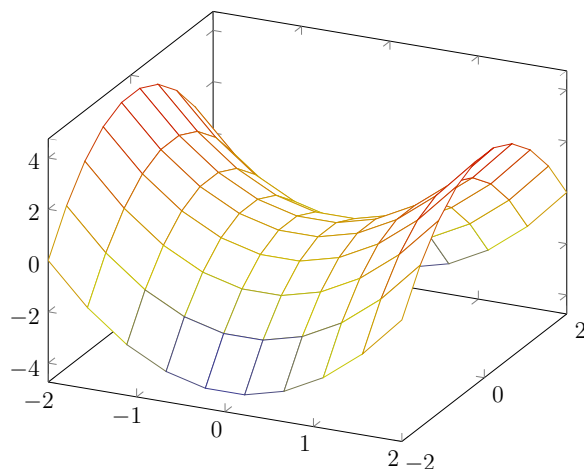
It's important to note that all maximums and minimums are *local*. To be an absolute maximum or minimum, you have to know the heights and trends of the entire function.

To classify critical points as maximums or minimums, we look at the second derivative. The point was called a minimum if $f''(x_0) > 0$ and it was called a maximum if $f''(x_0) < 0$. I like the mnemonic, "concave up (+) is like a cup; concave down (-) is like a frown."

For functions of two variables, $z = f(x, y)$, we do something similar.

Definition 22.1: A point (a, b) is a **critical point** of $z = f(x, y)$ if the gradient, ∇f , is the zero vector or if it is undefined.

Critical points in three dimensions can be maximums, minimums, or saddle points. A **saddle point** mixes a minimum in one direction with a maximum in another direction, so it's neither (see the image below).



Once a point is identified as a critical point, we want to be able to classify it as one of the three possibilities. Like you did in calculus, you will look at the second derivative. In higher dimensions, this is the determinant of a matrix containing all possible second derivatives, denoted as d . This matrix is called the **Hessian matrix**.

$$d = \det \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}$$

A determinant is a process that takes in a matrix and produces a number. For a 2×2 matrix, it means multiplying the diagonal entries and subtracting the product of the off diagonal entries:

$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

The critical point is classified by the value of d .

- If $d > 0$ and $f_{xx}(a, b) > 0$, then the point is a **(local) minimum**. For f_{xx} we can think, “concave up (+) is like a cup.”
- If $d > 0$ and $f_{xx}(a, b) < 0$, then the point is a **(local) maximum**. For f_{xx} we can think, “concave down (-) is like a frown.”
- If $d < 0$, then the point is a **saddle point**.
- If $d = 0$, then we call the critical point **degenerate** because the test was inconclusive.

For degenerate points, often there is a clever argument one can make that let’s you classify it. In Example 22.4, this is precisely what happens.

Example 22.2: Find and classify all critical values for the following function.

$$f(x, y) = xy - 2x - 2y - x^2 - y^2$$

First, we need to find the zeros of the partial derivatives. Those partials are

- $f_x(x, y) = y - 2 - 2x$
- $f_y(x, y) = x - 2 - 2y$

Set both of these partial derivatives to zero.

- $0 = y - 2 - 2x$
- $0 = x - 2 - 2y$

Then we solve the system of equations.

$$x = 2 + 2y \implies y = 2 + 2(2 + 2y) \implies y = 2 + 4 + 4y$$

Then $-3y = 6$ gives us that $y = -2$. We can plug in to find x

$$x = 2 + 2(-2) = -2$$

The solution is $\boxed{(-2, -2)}$. That is our critical point.

Now, we need to classify it. Let's find the second partial derivatives:

- $f_{xx}(x, y) = -2$
- $f_{yy}(x, y) = -2$
- $f_{xy}(x, y) = 1$

Then

$$d = (-2)(-2) - 1 = 3$$

Since $d = 3 > 0$ and $f_{xx} = -2 < 0$, then we have a $\boxed{\text{local maximum}}$.

Example 22.3: Find and classify all critical values for the following function.

$$f(x, y) = x^3 - 12xy - 8y^3$$

First, we need to find the zeros of the partial derivatives. Those partials are

- $f_x(x, y) = 3x^2 - 12y$
- $f_y(x, y) = -12x - 24y^2$

Set both of these partial derivatives to zero.

- $y = (1/4)x^2$
- $x = -2y^2$

Next, we solve the system of equations.

$$y = (1/4)(-2y^2)^2 \implies y = y^4$$

If $y = 1$, then $x = -2$. If $y = 0$, then $x = 0$. The critical points $\boxed{(-2, 1)}$ and $\boxed{(0, 0)}$.

We now calculate the second derivatives to classify the critical point.

- $f_{xx}(x, y) = 6x$
- $f_{yy}(x, y) = -48y$
- $f_{xy}(x, y) = -12$

Then

$$d = (6x)(-48y) - (12)^2 = -288xy - 144$$

Now, let's determine d for the point $(-2, 1)$.

$$d(-2, 1) = 576 - 144 = 432 > 0$$

Since $d(-2, 1) = 432 > 0$ and $f_{xx}(-2, 1) = 6(-2) = -12 < 0$, then $(-2, 1)$ a $\boxed{\text{local maximum}}$.

$$d(0, 0) = 0 - 144 < 0$$

Thus, $(0, 0)$ is a $\boxed{\text{saddle point}}$.

Example 22.4: Find and classify all critical values for the following function.

$$f(x, y) = \sqrt{x^2 + y^2}$$

As before, we begin by finding the partial derivatives:

- $f_x(x, y) = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = x(x^2 + y^2)^{-1/2} = \frac{x}{\sqrt{x^2 + y^2}}$
- $f_y(x, y) = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2y = y(x^2 + y^2)^{-1/2} = \frac{y}{\sqrt{x^2 + y^2}}$

Set both of these equations equal to zero.

- $0 = \frac{x}{\sqrt{x^2 + y^2}} \implies x = 0$
- $0 = \frac{y}{\sqrt{x^2 + y^2}} \implies y = 0$

Hence, our only critical point is $(0, 0)$.

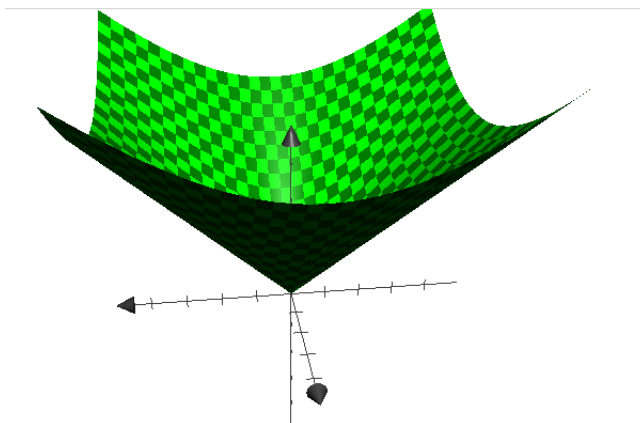
Let's now calculate the second derivatives:

- $f_{xx}(x, y) = x \cdot \left(-\frac{1}{2}\right)(x^2 + y^2)^{-3/2} \cdot 2x + 1 \cdot x(x^2 + y^2)^{-1/2} = \frac{-x^2 + x(x^2 + y^2)}{(x^2 + y^2)^{3/2}}$
- $f_{xy}(x, y) = x \cdot \left(-\frac{1}{2}\right)(x^2 + y^2)^{-3/2} \cdot 2y + 0 \cdot x(x^2 + y^2)^{-1/2} = \frac{-xy}{(x^2 + y^2)^{3/2}}$
- $f_{yy}(x, y) = y \cdot \left(-\frac{1}{2}\right)(x^2 + y^2)^{-3/2} \cdot 2y + 1 \cdot y(x^2 + y^2)^{-1/2} = \frac{-y^2 + y(x^2 + y^2)}{(x^2 + y^2)^{3/2}}$

All of the equations above are undefined at $(0, 0)$. So the critical point is **degenerate**.

How can we determine if it is a maximum, a minimum, or a saddle point? Let's return to the original function $f(x, y) = \sqrt{x^2 + y^2}$. Notice that it is always positive. So at $(0, 0)$, it's at its lowest value. Hence, the point is a minimum.

If we graph the function, we see that this is correct.



Summary of Ideas

- Critical points are those points (x, y) such that $\nabla f(x, y) = 0$ or if ∇f is undefined.
- They are classified as local maximums, minimums and saddles using the determinant of the **Hessian matrix**.

$$d = \det \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{yx}(a, b) & f_{yy}(a, b) \end{vmatrix}$$

- We classify according the following rules:
 - If $d > 0$ and $f_{xx}(a, b) > 0$, then the point is a **(local) minimum**.
 - If $d > 0$ and $f_{xx}(a, b) < 0$, then the point is a **(local) maximum**.
 - If $d < 0$, then the point is a **saddle point**.
 - If $d = 0$, then the point is **degenerate** and it could be anything.