21. Lecture 21

Objectives

- ☐ I can use the chain rule to find the derivative of a function with respect to a parameter like time.
- ☐ I know to write the result of the derivative in terms of the parameter(s) only.

Up to this point, we have focused on partial derivatives of a function $f(x, y)$ based on the variables x and y. In practice, however, the independent variables actually dependent on some "hidden variable" like time. We call these "hidden variables," **parameters**.

In this section, we will discuss how to take the derivative of the dependent variable with respect to some parameter. We will also introduce some new notation for partial derivatives.

Let's begin our discussion by motivating it with an example. On the last quiz, you were given the Cobb-Douglas production function

$$P(L, K) = AL^{\alpha}K^{\beta}$$

which models how a country's yearly production P is related to the number of labor hours L and the amount of invested capital K. Put in layman's terms, *the amount of money a country makes annually depends on how much people work (work hours) and what tools they have available to them (invested capital)*. The constants A, α and β relate to how efficiently a country can use its labor and capital investment. For example, if the population is filled with unproductive workers (maybe due to lack of education), then α will be low. If there aren't enough people who know how to use the capital investment (for example, if there are more busses than bus drivers), then β will be low. If there is a lot of corruption or crime, A will be low.

When we look at a partial derivative like P_L, we are seeing how much P changes if we change L by one unit. This is important for policy-makers to discern.

Policy makers may also want to understand how much production changes with time, $\frac{dP}{dt}$. For factors beyond anyone's control, L and K will change with time. For example, an aging population will cause L to go down with time. If a country can't continue to re-invest in its capital, like Cuba due to embargoes, the capital they have will slowly go down with time (tools rust, machines break, etc). When countries want to understand trends with time, P_L and P_K do not provide enough information.

To understand the derivative of P with respect to t, we will need to use the chain rule. Let us remind ourselves of how the chain rule works with two dimensional functions. If we are given the function $y = f(x)$, where x is a function of time: $x = g(t)$. Then the derivative of y with respect to t is the derivative of y with respect to x multiplied by the derivative of x with respect to t

$$\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt} = f'(g(t))g'(t)$$

The technique for higher dimensions works similarly. The only difficulty is that we need to consider <u>all</u> the variables dependent on the relevant parameter (time t).

Suppose $z = f(x, y)$ and $x = g(t), y = h(t)$. Based on the one variable case, we can see that dz/dt is calculated as

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

In this context, it is more common to see the following notation.

$$f_x = \frac{\partial f}{\partial x}$$

The symbol ∂ is referred to as a "partial," short for partial derivative.

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The procedures for calculating the derivative with respect to a parameter will be the following:

(1) Find the partial derivatives, f_x and f_y.
(2) Plug in the "inside functions" (i.e. plug in $g(t)$ for x when $x = g(t)$) into the partial derivatives
(3) Find the derivatives of the "inside functions."
(4) Calculate

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

Notice that we plug in the "inside functions." Why? Because it's less confusing to have our final answer completely in terms of the parameter variable. If we neglected this step, we would have an expression with x, y, and t.

Example 21.1: Use the chain rule to find dz/dt for

$$z = \ln(4x + y), \quad x = 5t^4, y = \frac{1}{t}$$

Solution 21.2: We begin by finding all the necessary partial derivatives. We put everything in terms of t by plugging in the functions for x and y. This gives us the following equations:

$$\frac{\partial z}{\partial x} = \frac{-4}{4x+y} = \frac{-4}{4[5t^4] + [1/t]} = \frac{-4}{\frac{20t^4+1}{t}} = \frac{-4t}{20t^4+1}$$

$$\frac{\partial z}{\partial y} = \frac{-1}{4x+y} = \frac{-1}{(4[5t^4] + [1/t])} = \frac{-1}{\frac{20t^4+1}{t}} = \frac{-t}{20t^4+1}$$

$$\frac{dx}{dt} = 20t^3$$

$$\frac{dy}{dt} = \frac{-1}{t^2}$$

Now, we just plug into the formula

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

to get

$$\frac{dz}{dt} = \frac{-4t}{20t^4+1} \cdot 20t^3 + \frac{-t}{20t^4+1} \cdot \frac{-1}{t^2}$$

$$\boxed{\frac{dz}{dt} = \frac{-80t^4}{20t^4+1} + \frac{t}{20t^6+t^2} = \frac{t-80t^6}{20t^6+t^2}}$$

Example 21.3: The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t second is given by $x = \sqrt{1+t}$, $y = 2 + \frac{t}{3}$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bugs path after 3 seconds?

Solution 21.4: At first glance, it may appear that we lack enough information to answer this question. Surprisingly, we have just enough.

Because the symbols are different than above, it's helpful to begin by writing out the derivative of T with respect to time:

$$\frac{dT}{dt} = T_x \cdot \frac{dx}{dt} + T_y \cdot \frac{dy}{dt}$$

Now, we wish to determine this when $t = 3$. That means $x = \sqrt{1+3} = \sqrt{4} = 2$ and $y = 2 + \frac{3}{3} = 3$. Hence, we want to calculate all these derivatives with the assumption that $x = 2, y = 3$, and $t = 3$.

$$\begin{aligned}
&T_x(2,3) = 4 \text{ (this was given)}\\
&T_y(2,3) = 3 \text{ (this was also given)}\\
&\frac{dx}{dt}(3) = \frac{1}{2}(1+t)^{-1/2}\Big|_{t=3} = \frac{1}{2}(1+3)^{-1/2} = \frac{1}{2\sqrt{4}} = \frac{1}{4}\\
&\frac{dy}{dt}(3) = \frac{1}{3}
\end{aligned}$$

Now we plug into our expression $\frac{dT}{dt} = T_x \cdot \frac{dx}{dt} + T_y \cdot \frac{dy}{dt}$ and get

$$\frac{dT}{dt} = (4) \cdot \left(\frac{1}{4}\right) + (3) \cdot \left(\frac{1}{3}\right) = \boxed{2}$$

Thus, the temperature rises two degrees on the bug's path after 3 seconds.

Summary of Ideas

- Sometimes x and y are functions of one or more parameters. We may find the derivative of a function with respect to that parameter using the **chain rule**.
- The formula for calculating such a derivative is

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad \text{and} \qquad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$