16. LECTURE 16

Objectives

- ☐ I understand how to find the rate of change in any direction.
- ☐ I understand in what direction the maximum rate of change happens.

So far, we've learned the definition of the gradient vector and we know that it tells us the direction of steepest ascent. Its magnitude indicates the rate of change of the dependent variable in the direction of the gradient.

A natural question to ask is, "What is the rate of change of the dependent variable in the direction of an arbitrary vector $\vec{v}$?" In other words, how fast does the surface ascend in the direction of $\vec{v}$?

The rate of change in the direction of $\vec{v}$ is called the *directional derivative*. That is the *scalar projection of the gradient onto $\vec{v}$.*

Definition 16.1: The **directional derivative**, denoted $D_v f(x, y)$, is a derivative of a multivariable function in the direction of a vector $\vec{v}$. It is the scalar projection of the gradient onto $\vec{v}$.

$$D_v f(x,y) = \text{comp}_v \nabla f(x,y) = \frac{\nabla f(x,y) \cdot \vec{v}}{|\vec{v}|}$$

It's best to understand concepts with a picture. So let's draw one. Consider the function

$$f(x,y) = x^2 - y^2.$$

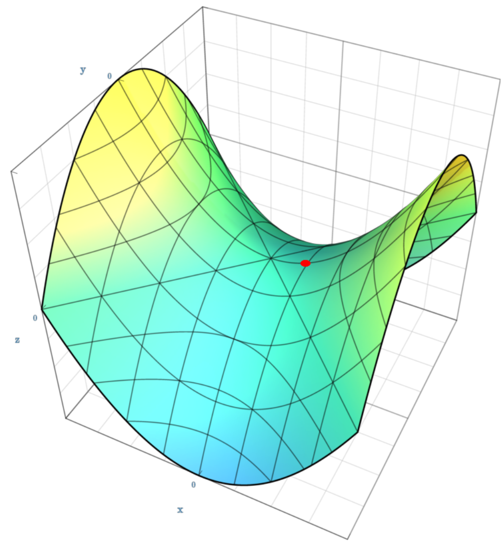
The gradient of f is

$$\nabla f(x,y) = \begin{pmatrix} 2x \\ -2y \end{pmatrix}.$$

At the point $(1, 0)$, the direction of steepest ascent is

$$\nabla f(1,0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

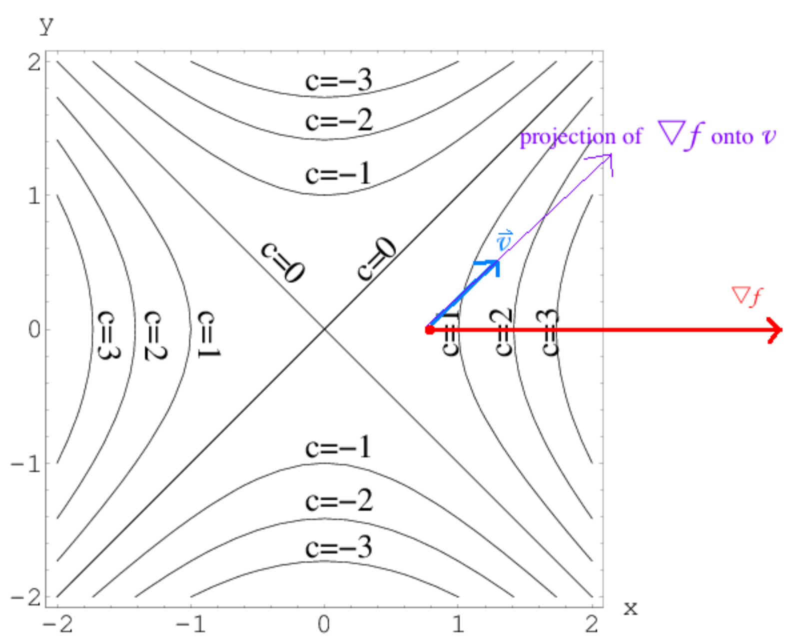
In that direction, f has a slope of $|\nabla f(1,0)| = \sqrt{(2)^2} = 2$.



What is the slope at $(1,0)$ in the direction of

$$\vec{v} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}?$$

$$D_v f(x,y) = \text{comp}_v \nabla f(1,0) = \frac{\begin{pmatrix} 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}}{\sqrt{0^2+1^2}} = \frac{1}{1} = 1$$



Let's look at some examples.

Example 16.2: Find the directional derivative of

$$f(x, y) = \frac{x}{x^2 + y^2}$$

in the direction of $\vec{v} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ at the point $(1, 2)$.

First, we find the gradient.

$$f_x(x, y) = \frac{d}{dx}\left(\frac{x}{x^2 + y^2}\right)$$

$$= \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{d}{dy}\left(\frac{x}{x^2 + y^2}\right)$$

$$= \frac{-2xy}{(x^2 + y^2)^2}$$

The gradient is then

$$\nabla f(1, 2) = \begin{pmatrix} \frac{4-1}{(4+1)^2} \\ \\ -\frac{4}{(4+1)^2} \end{pmatrix} = \begin{pmatrix} \frac{3}{25} \\ \\ -\frac{4}{25} \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

We now find the magnitude of $\vec{v}$. We get

$$|\vec{v}| = \sqrt{9 + 25} = \sqrt{34}$$

The directional derivative is then

$$D_v f(1, 2) = \frac{\nabla f(1, 2) \cdot \vec{v}}{|\vec{v}|} = \frac{1}{25\sqrt{34}}\begin{pmatrix} 3 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \frac{1}{25\sqrt{34}}(9 - 20) = \boxed{-\frac{11}{25\sqrt{34}}}$$

Example 16.3: Find the directional derivative of

$$f(x, y, z) = \sqrt{xyz}$$

in the direction of $\vec{v} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$ at the point $(3, 2, 6)$.

First, we find the partial derivatives to define the gradient.

$$f_x(x,y,z) = \frac{yz}{2\sqrt{xyz}}$$
$$f_y(x,y,z) = \frac{xz}{2\sqrt{xyz}}$$
$$f_z(x,y,z) = \frac{xy}{2\sqrt{xyz}}$$

The gradient is

$$\nabla f(3,2,6) = \begin{pmatrix} \frac{12}{2(6)} \\ \\ \frac{18}{2(6)} \\ \\ \frac{6}{2(6)} \end{pmatrix} = \begin{pmatrix} 1 \\ \\ \frac{3}{2} \\ \\ \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

The magnitude of $\vec{v} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$ is

$$|\vec{v}| = \sqrt{1+4+4} = 3$$

Therefore, the directional derivative is

$$D_v f(3,2,6) = \frac{\nabla f(3,2,6)\cdot \vec{v}}{|\vec{v}|} = \frac{1}{3(2)}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} = \frac{1}{6}(-2-6+2) = \boxed{-1}$$

The next natural question is:

In what direction is the derivative maximum?

As we just saw, the directional derivative is calculated by taking the scalar projection of ∇f onto a vector $\vec{v}$. Define θ be the angle between $\vec{v}$ and ∇f. Then,

$$\frac{\nabla f \cdot \vec{v}}{|\vec{v}|} = \frac{|\nabla f||\vec{u}|\cos(\theta)}{|\vec{v}|} = |\nabla f|\cos(\theta)$$

This is maximized if $\theta = 0$. From this, we know the following:

- The maximum rate of change (the largest directional derivative) is $|\nabla f|$.
- This occurs when $\vec{v}$ is parallel to ∇f, i.e. when they point in the same direction.

That makes sense since ∇f is the vector pointing toward steepest ascent, so it should be the direction with the largest derivative.

Observe, also that...

- No change occurs when $\theta = 90°$ or when $\theta = -90°$. In other words, directions perpendicular to the gradient are constant height.
- The rate of steepest *descent* happens when $\theta = 180°$. It's rate of change is $-|\nabla f|$,

Let's look at two examples.

Example 16.4: Find the maximum rate of change of f at the given point and the direction in which it occurs.

$$f(s,t) = te^{st}, \quad \text{at } (0,2)$$

The maximum rate of change is $|\nabla f(0,2)|$. Let's first find the gradient.

$$\nabla f = \begin{pmatrix} te^{st} \\ ste^{st} + e^{st} \end{pmatrix}$$

Then

$$|\nabla f(0,2)| = \sqrt{(2)^2 + 1^2} = \boxed{\sqrt{5}}$$

The direction is

$$\nabla f(0,2) = \boxed{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

Remark 16.5: For this problem, it may not have been clear which component was the first and which was the second since s and t are atypical variables. For clues about the order, look at how the ordered pairs are defined in the function. It was written as "$f(s,t)$," which tells us our gradient vector should be $\begin{pmatrix} f_s \\ f_t \end{pmatrix}$.

Example 16.6: Find the maximum rate of change of f at the given point and the direction in which it occurs.

$$f(x,y,z) = \sqrt{x^2 + y^2 + z^2}, \quad (3,6,-2)$$

As above, the maximum rate of change is $|\nabla f(3,6,-2)|$.

$$\nabla f(x,y,z) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} \\ \\ \frac{y}{\sqrt{x^2+y^2+z^2}} \\ \\ \frac{z}{\sqrt{x^2+y^2+z^2}} \end{pmatrix}$$

Then

$$\nabla f(3,6,-2) = \begin{pmatrix} \frac{3}{7} \\ \\ \frac{6}{7} \\ \\ -\frac{2}{7} \end{pmatrix}$$

The $|\nabla f(3,6,-2)| = 1/7\sqrt{9+36+4} = \boxed{1}$

The direction is

$$\boxed{\begin{pmatrix} \frac{3}{7} \\ \\ \frac{6}{7} \\ \\ -\frac{2}{7} \end{pmatrix}}$$

Summary of Ideas: Lecture 16

- The **gradient vector** of a function f, denoted ∇f or $grad(f)$, is a vectors whose entries are the partial derivatives of f.
$$\nabla f(x, y) = \left(\ f_x(x, y), f_y(x, y) \ \right)$$
It is the generalization of a derivative in higher dimensions.
- The gradient points in the direction of **steepest ascent**.
- The **directional derivative**, denoted $D_v f(x, y)$, is a derivative of a $f(x, y)$ in the direction of a vector $\vec{v}$. It is the scalar projection of the gradient onto $\vec{v}$.
$$D_v f(x, y) = \text{comp}_v \nabla f(x, y) = \frac{\nabla f(x, y) \cdot \vec{v}}{|\vec{v}|}$$
This produces a vector whose magnitude represents the rate a function ascends (how steep it is) at point (x, y) in the direction of $\vec{v}$.
- The **maximum directional derivative** is always $|\nabla f|$.
- This happens in the direction of ∇f