**Objectives**

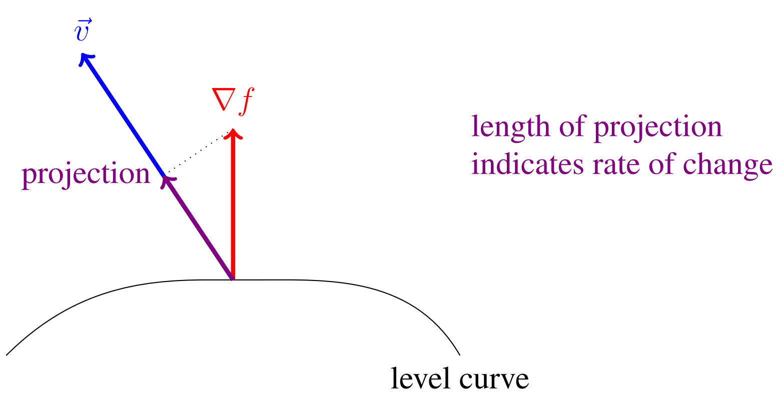
- ☐ I can calculate the dot product of two vectors and interpret its meaning.
- ☐ I can find the projection of one vector onto another one.

In the last few lectures, we've learned that

- vectors perpendicular to a level curve point in the direction of steepest ascent or descent;
- the gradient vector, $\nabla f$, points in the direction of steepest ascent, and its negative, $-\nabla f$ points in the direction of steepest descent; and
- the magnitude of the gradient, $|\nabla f|$ tells us the rate at which the dependent variable increases in the direction of $\nabla f$.

This last point can be rephrased as the derivative of $f$ in the direction of $\nabla f$ is $|\nabla f|$.

What about in other directions? What's the derivative of $f$ a point in a direction $\vec{v}$, which different from $\nabla f$ and $-\nabla f$? The answer to this question is: *the projection of $\nabla f$ onto $\vec{v}$ and find the magnitude of that projection.* This is depicted below.




To understand this, we'll need to first learn about projections. Before we begin, let's take a moment to recall the following facts:

- The **dot product** of two vectors, $\vec{v}$ and $\vec{w}$, is obtained by taking the sum of the product of the entries

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_1 w_2 + v_3 w_3 + \dots$$

- A **unit vector** is a vector with magnitude equal to one.

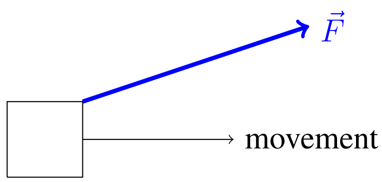
- To find a unit vector $\vec{u}$ that points in the *same* direction as an arbitrary vector $\vec{v}$, we must divide by the vector's magnitude. That is,

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \begin{pmatrix} \frac{v_1}{|\vec{v}|} \\ \frac{v_2}{|\vec{v}|} \\ \frac{v_3}{|\vec{v}|} \\ \vdots \end{pmatrix}$$

15.1. **Dot Products.** Because vectors describe movement, we might ask the question:

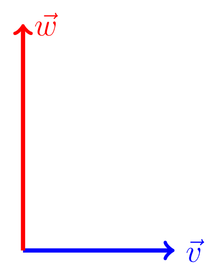
"To what extent do two vectors move together?"

Consider a box being dragged horizontally by a force applied at an angle. Both its movement and its force are described using vectors. We are interested in the amount these two vectors share the same direction.

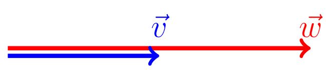



We will use this notion to understand the **dot product**. This will be a scalar value.[8] Before we go into the formulas, let's talk a little bit about this idea of moving together.

If I have two perpendicular vectors, they don't move in the same direction at all. These vectors are referred to as **orthogonal**. We will represent the amount they move together with 0. That is, $\vec{w} \cdot \vec{v} = 0$.




Parallel vectors move entirely in the same direction. We will represent the amount they move together with the product of their magnitudes. That is, $\vec{w} \cdot \vec{v} = |\vec{w}||\vec{v}|$.




From these two examples, we can see that the angle between the two vectors plays a part. Since $0°$ corresponds to 1 and $90°$ corresponds to 0, we can deduce that our equation for the dot product is

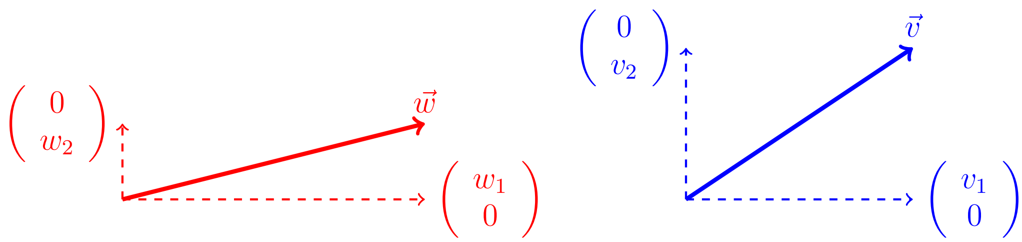
$$\vec{w} \cdot \vec{v} = |\vec{w}||\vec{v}| \cos\theta$$

[8] In fact, every notion of a measurement will be a scalar value.

where $\theta$ is the angle between them. From a few lectures previous, we know there exists another formula. For vectors $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$,

$$\vec{w} \cdot \vec{v} = w_1 v_1 + w_2 v_2$$

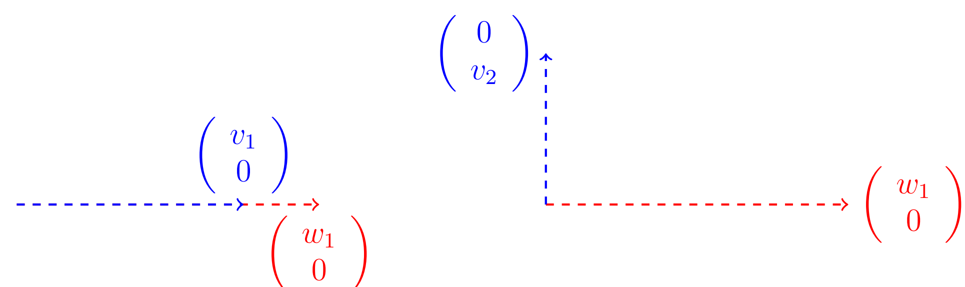
How might these two formulas be related? Let's think about the components of each vector.



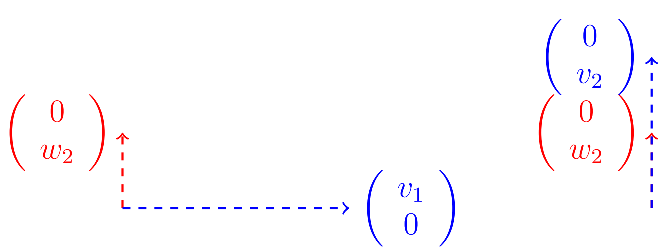
Then we can FOIL out the separate components.

$$\vec{w}\cdot\vec{v} = \begin{pmatrix} w_1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ 0 \end{pmatrix}+\begin{pmatrix} w_1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ v_2 \end{pmatrix}+\begin{pmatrix} 0 \\ w_2 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ w_2 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ v_2 \end{pmatrix} = w_1v_1+0+0+w_2v_2$$

For the components that are parallel, we will get the product of the magnitudes. For the components that are perpendicular, we will get zero. Let's look at each pairing of vectors to see which is which.



$$\begin{pmatrix} u_1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = w_1 v_1 \qquad \begin{pmatrix} w_1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = 0$$



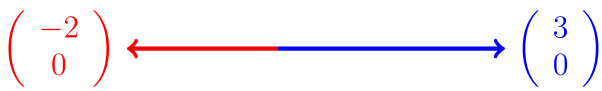
$$\begin{pmatrix} 0 \\ w_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = 0 \qquad \begin{pmatrix} 0 \\ w_2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = w_2 v_2$$

Basically, when we break up a vector by its respective components and look at the dot product, we see why these two equations agree. The perpendicular components will have a zero dot product while the parallel components will only look at the product.

What happens when two parallel components point in opposite directions? For example, when

$$\begin{pmatrix} -2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Then the dot product can be expressed as either $|-2||3|\cos 180° = -6$ and $(0) + (-2)(3) = -6$. Where did this $180°$ come from? If the vectors are parallel, then isn't the angle between them zero? No, it turns out the angle is measure when we place vectors from end to end. Remember, these objects have no location, so we can shift them anywhere.




15.2. **Projections.** With the use of dot products, we can talk about projections. There are two notions of a projection:

- a scalar projection
- a vector projection

The **scalar projection** of $\vec{a}$ onto $\vec{b}$ indicates the amount $\vec{a}$ moves in the particular direction of $\vec{b}$. It is denoted $\text{comp}_{\vec{b}}\vec{a}$. The formula is

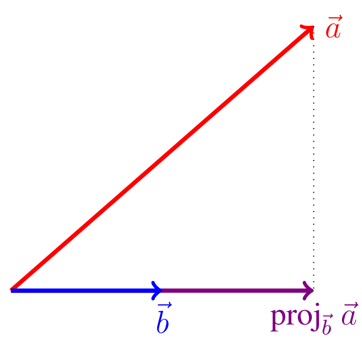
$$\text{comp}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

Why does this formula make sense? The dot product tells us the amount in which the two vectors move in the same direction. Because we only want a sense of how much $\vec{a}$ moves, we divide by the magnitude of $|\vec{b}|$.

The **vector projection** of $\vec{a}$ onto $\vec{b}$ is a vector representative of the amount $\vec{a}$ moves in the direction of $\vec{b}$. It is denoted $\text{proj}_{\vec{b}}\,\vec{a}$. The formula is $\text{comp}_{\vec{b}}\,\vec{a}$ times the unit vector in the direction of $\vec{b}$.

$$\text{proj}_{\vec{b}}\,\vec{a} = (\text{comp}_{\vec{b}}\,\vec{a})\frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\,\vec{b}$$

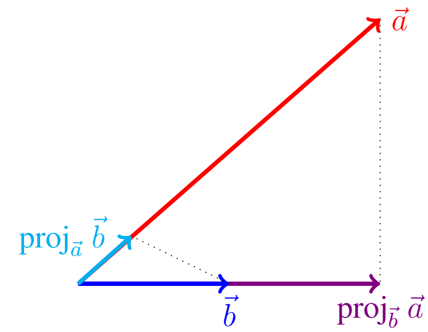
Notice that this is just the scalar projection multiplied by the unit vector in the direction of $\vec{b}$. This makes sense because the vector projection of $\vec{a}$ onto $\vec{b}$ should be a vector in the direction of $\vec{b}$ whose magnitude reflects the amount at which $\vec{a}$ points in the direction of $\vec{b}$.




One more thing to point out is that

$$\text{proj}_{\vec{b}}\,\vec{a} \neq \text{proj}_{\vec{a}}\,\vec{b}.$$

Let's look at a picture of the projection of $\operatorname{proj}_{\vec{a}} \vec{b}$ to illustrate this fact.




The magnitude of our projection is $\operatorname{comp}_{\vec{a}} \vec{b}$, so we can also deduce from this illustration that

$$\operatorname{comp}_{\vec{b}} \vec{a} \neq \operatorname{comp}_{\vec{a}} \vec{b}.$$

In other words, order matters!

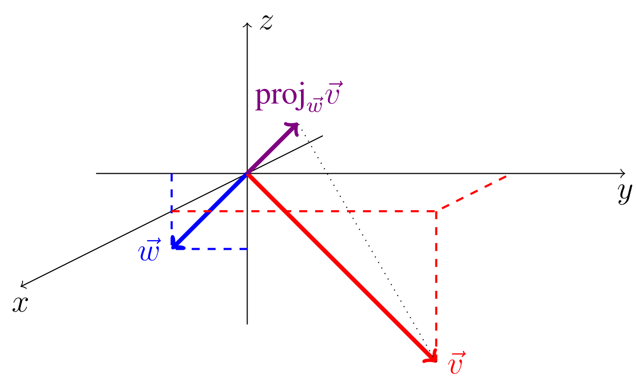
**Example 15.1:** Find the scalar and vector projections of $\vec{v}$ onto $\vec{w}$ where

$$\vec{v} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \text{ and } \vec{w} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

The question asks for $\vec{v}$ onto $\vec{w}$, so we're looking to find $\operatorname{comp}_{\vec{w}}\vec{v}$ and $\operatorname{proj}_{\vec{w}}\vec{v}$. When you sit down to memorize the formulas, remember that the vector you project onto will be the one that appears the most.

$$\operatorname{comp}_{\vec{w}}\vec{v} = \frac{\vec{v} \cdot \vec{w}}{|\vec{w}|} = \frac{(1)(0) + (3)(-1) + (-2)(-1)}{\sqrt{0^2 + (-1)^2 + (-1)^2}} = \boxed{-\frac{1}{\sqrt{2}}}$$

The scalar projection reflects the amount in which $\vec{v}$ travels in the direction of $\vec{w}$. If the value is negative, it tells me that the vector $\operatorname{proj}_{\vec{w}}\vec{v}$ is traveling in the opposite direction to $\vec{w}$.




Now, let's describe the vector pictured.

$$\text{proj}_{\vec{w}}\vec{v} = -\frac{1}{\sqrt{2}} \frac{\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}}{\sqrt{2}} = -\frac{1}{2} \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \boxed{\left( 0, \tfrac{1}{2}, \tfrac{1}{2} \right)}$$

**Summary of Ideas: Lecture 15**

- The **dot product** is a measurement of how much two vectors move together. We have two formulas for it.

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta \quad \text{and} \quad \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

- We can also use the dot product to measure the amount one vector ($\vec{a}$) moves in the direction of another vector ($\vec{b}$):

$$\text{comp}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

  This is called the **scalar projection**.
- To find the **vector projection**, we take the scalar projection and multiply by the unit vector we are projecting onto.

$$\text{proj}_{\vec{b}}\vec{a} = \text{comp}_{\vec{b}}\vec{a} \frac{\vec{b}}{|\vec{b}|}$$

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