Objectives

- ☐ I understand what a gradient vector is and what it tells you.
- ☐ I can use the gradient to identify important features of a surface like steepest ascent.

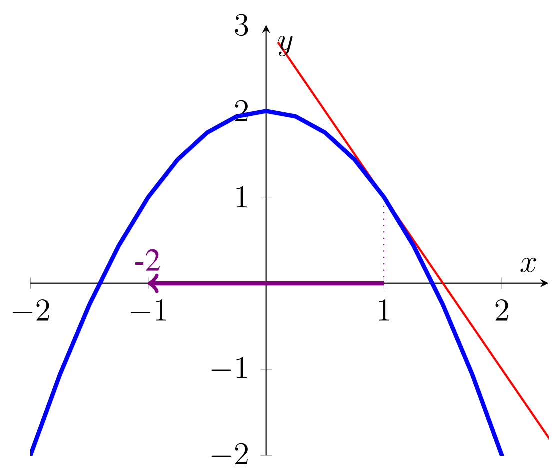
In lecture 11, we learned the direction of *steepest* change occurred perpendicularly to the level curves. But how much change can we see? What is this direction precisely? In this lecture, we will determine that direction.

The derivative of a function $f(x)$ at a point x is the slope of the tangent line of f at x. Loosely, we might say that it is the slope of f at that point x. As we will see, the derivative will also tell us a direction of steepest ascent.

When we are thinking of a two-dimensional function, there are only two choices–left or right–and the sign of the derivative tells us this direction.

For example, if you're given the function $f(x) = -x^2+2$, you know the derivative of f at $x = 1$ will be -2. If we represented this derivative as a vector (an arrow) on the x-axis, then

(1) it points in the direction of **steepest ascent** and
(2) it's length represents the steepness of the incline.



For higher dimensions, the approach is similar. We want to construct vectors that point in the direction of steepest ascent whose length represents the steepness of the incline. These vectors are constructed from partial derivatives! This vector is called the gradient vector.

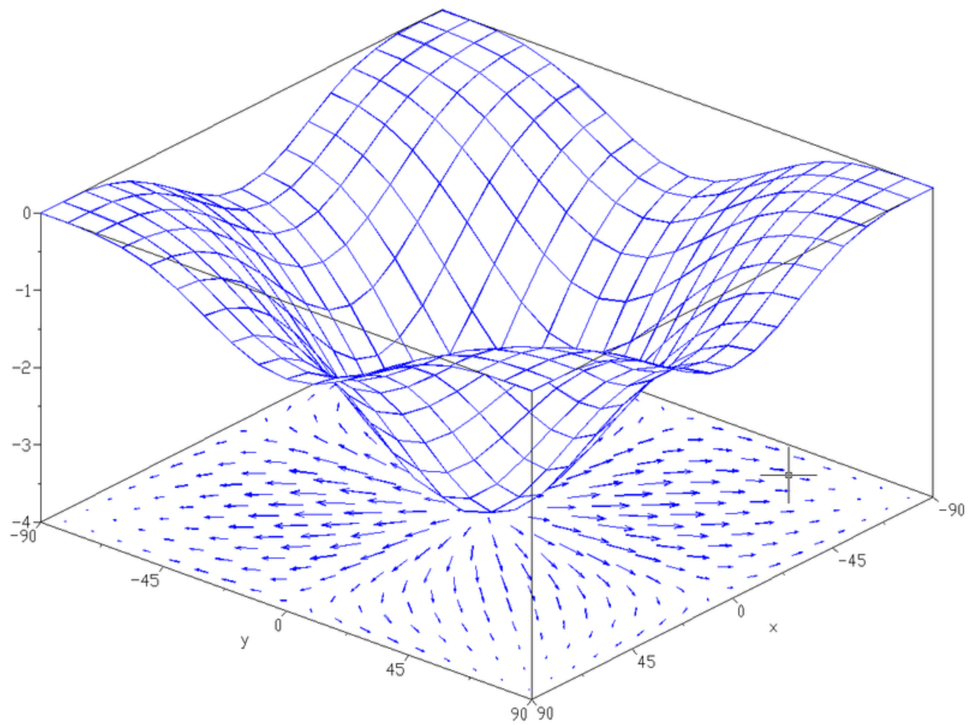
Definition 14.1: The **gradient vector** of a function f, denoted ∇f or $grad(f)$, is a vectors whose entries are the partial derivatives of f. That is,

$$\nabla f(x,y) = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$$

For higher dimensions, we have

$$\nabla f(x_1, \ldots, x_n) = \begin{pmatrix} f_{x_n} \\ \vdots \\ f_{y_n} \end{pmatrix}$$

Here is a picture of a three dimensional surface. The arrows at the bottom represent the gradient vectors. They each point in the direction where it is steepest. Longer the vector (the greater its magnitude), the steeper the surface is at that point.



As we will see in the examples below, the gradient vector is always perpendicular to some level curve.

Example 14.2: Find the gradient vector of

$$f(x, y) = x^2 + y^2$$

What are the gradient vectors at $(1, 2)$, $(2, 1)$ and $(0, 0)$?

Solution 14.3: We begin with the formula.

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \boxed{\begin{pmatrix} 2x \\ 2y \end{pmatrix}}$$

Now, let us find the gradient at the following points.

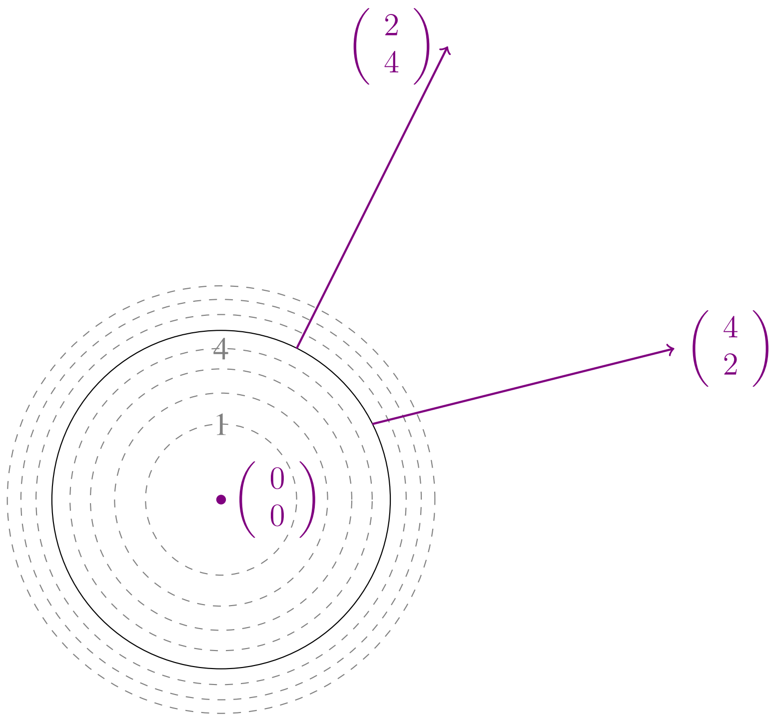
- $\nabla f(1, 2) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$
- $\nabla f(2, 1) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

- $\nabla f(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

How steep are these gradients? Let's calculate those lengths.

- $|\nabla f(1,2)| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$
- $|\nabla f(2,1)| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$
- $|\nabla f(0,0)| = \sqrt{0^2 + 0^2} = \sqrt{0} = 0$

Now, let's verify that these vectors are perpendicular to their level curve. Notice that $(1,2)$ corresponds to the level curve of height $f(1,2) = 1^2 + 2^2 = 5$. Similarly, $f(2,1) = 5$.



Example 14.4: Find the gradient vector of

$$f(x,y) = 2xy + x^2 + y$$

What are the gradient vectors at $(1,1)$, $(0,-1)$ and $(0,0)$?

Solution 14.5:

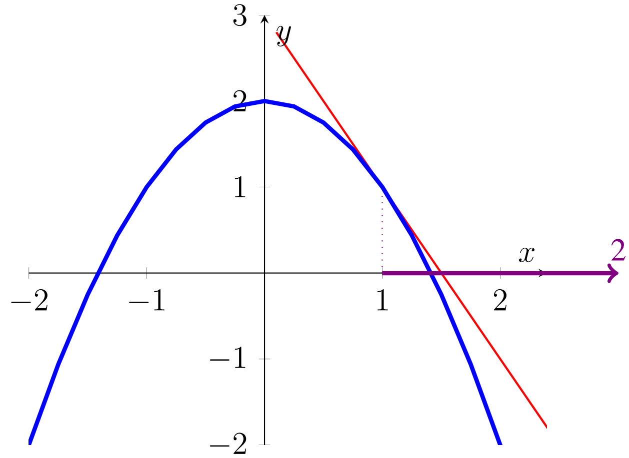
$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \boxed{\begin{pmatrix} 2y + 2x \\ 2x + 1 \end{pmatrix}}$$

Now, let us find the gradient at the following points.

- $\nabla f(1,1) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$
- $\nabla f(0,-1) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

- $\nabla f(0,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Suppose we want to identify the direction of steepest descent? In the two-dimensional case, we simply switch the sign of the derivative. For example, if we return to the function $f(x) = -x^2 + 2$, the arrow of steepest descent points in the positive direction and is length 2.



For higher dimensions, it is the same. That is, $-\nabla f$ always points in the direction of steepest descent.

Example 14.6: Find the direction of steepest ascent

$$f(x,y) = 2xy + x^2 + y$$

at $(1,1)$, $(0,-1)$ and $(0,0)$? How steep are they?

Solution 14.7: The directions of steepest ascent are

- $-\nabla f(1,1) = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$
- $-\nabla f(0,-1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$
- $-\nabla f(0,0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

The derivatives are

- $|-\nabla f(1,1)| = \sqrt{(-4)^2(-3)^2} = \sqrt{25} = 5$
- $|-\nabla f(0,-1)| = \sqrt{2^2(-1)^2} = \sqrt{5}$
- $|-\nabla f(0,0)| = \sqrt{0^2 + (-1)^2} = \sqrt{1} = 1$

The next obvious question is this:

How do we find how steep a surface is in other directions? To do that, we will need to understand *projections*. We will use the dot product to project the gradient vector in a direction to understand how steeply the surface is increasing/decreasing in that direction.

Summary of Ideas: Lecture 14

- The **gradient vector** of a function f, denoted ∇f or $grad(f)$, is a vectors whose entries are the partial derivatives of f.

$$\nabla f(x,y) = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix}$$

  It is the generalization of a derivative in higher dimensions.
- The gradient points in the direction of **steepest ascent**.
- $-\nabla f$ points in the direction of **steepest descent**.

PENN STATE UNIVERSITY, UNIVERSITY PARK, PA 16802