

## 13. LECTURE 13

### Objectives

- I can use Clairaut's Theorem to make my calculations easier.
- I can take higher derivatives.
- I can check if a function is a solution to a partial differential equation.

Higher order derivatives are calculated as you would expect. We still use subscripts to describe the second derivative, like  $f_{xx}$  and  $f_{yy}$ . These describe the concavity of  $f$  in the  $x$  and  $y$  direction, just like in Math 110.

Interestingly, we can get mixed second derivatives like  $f_{xy}$  and  $f_{yx}$ . These tell us how  $f_x$  changes with  $y$  and how  $f_y$  changes with  $x$ . Even more interesting is the fact that these are both equal!

**Theorem 13.1. (Clairaut's Theorem)**

$$f_{xy} = f_{yx}$$

What's the point of knowing this theorem? It means that you can switch the order of derivatives based on whatever would be easiest.

Clairaut's Theorem extends to higher derivatives. If we were looking at taking two derivatives with respect to  $x$  and one with respect to  $y$ , we would have three possible ways to do this

$$f_{yxx} = f_{xyx} = f_{xxy}$$

**Example 13.2:1.** Find  $f_{xxx}$ ,  $f_{xyx}$  for

$$f(x, y) = (2x + 5y)^7$$

**Solution 13.3:** Let's begin by finding  $f_x$  and use that to find  $f_{xx}$  and  $f_{xxx}$

$$f_x = 2[7(2x + 5y)^6]$$

Remember that  $5y$  is just treated as a constant. Notice that we could work towards finding  $f_{xyx}$  by finding  $f_{xy}$  from the above equation. If we use Clairaut's Theorem, however, we can skip a step by calculating  $f_{xxy}$  instead. Now, let's calculate  $f_{xx}$ .

$$f_{xx} = 2 \cdot 2[7 \cdot 6(2x + 5y)^5] = 168(2x + 5y)^5$$

Using  $f_{xx}$ , we can find  $f_{xxx}$  and  $f_{xxy}$ . They are

$$f_{xxx} = 2 \cdot 168[5(2x + 5y)^4] = 1680(2x + 5y)^4$$

and

$$f_{xyx} = f_{xxy} = 5 \cdot 168[5(2x + 5y)^4] = 4200(2x + 5y)^4$$

**Example 13.4:2.** Find  $f_{xyz}$  for

$$f(x, y, z) = e^{xyz^2}$$

**Solution 13.5:** This is a good example to pay close attention to because it illustrates how complicated these partial derivatives can get.

Let's first find  $f_x$ . It is

$$f_x = yz^2 e^{xyz^2}$$

Notice the coefficients. Because  $y$  and  $z$  are treated as constants, they need to be brought out front by the chain rule. For the next derivative, we will have to use the product rule. What does this tell us? It tells us that it's probably better to take  $f_z$  first since we won't get that pesky  $z^2$ .

$$f_z = 2zxy e^{xyz^2}$$

Notice that taking the derivative with respect to  $x$  or  $y$  next will result in the same amount of work. Let's just pick  $x$  next.

$$f_{zx} = (2zxy)(yz^2 e^{xyz^2}) + (2zy)(e^{xyz^2}) = 2xy^2 z^3 e^{xyz^2} + 2zy e^{xyz^2}$$

The parentheses are in place to indicate how I broke up the variables to take the derivatives. Now let's calculate the last derivative, the partial derivative with respect to  $y$ .

$$f_{zxy} = (2z)(e^{xyz^2}) + (2zy)(xz^2 e^{xyz^2}) + (2xy^2 z^3)(xz^2 e^{xyz^2}) + (4xyz^3)(e^{xyz^2})$$

After we simplify, we get the final answer

$$f_{zxy} = 2ze^{xyz^2} [1 + 3xyz^2 + x^2 y^2 z^4]$$

You may have heard of **partial differential equations**. These are equations that use derivatives of an unknown function as variables. The goal is to try to figure out the original function. For example, our understanding of waves is based on partial differential equations. Specifically, we look at something called **the wave equation**

$$u_{tt} = a^2 u_{xx}.$$

#### Learn More About This

Where does the wave equation come from, you ask? You can find out more about the wave equation and how it is derived by watching this video: [https://www.youtube.com/watch?v=ck-r\\_qmNNG0](https://www.youtube.com/watch?v=ck-r_qmNNG0).

Towards the end of this course, we'll discuss how to solve systems of differential equations. If time permits, we will discuss some partial differential equations like the wave equation. If you'd like to learn more about differential equations, I recommend the videos made by the Khan Academy. You can find those here: <https://www.khanacademy.org/math/differential-equations>

**Example 13.6:3.** Determine if  $u = x^2 + (at)^2$  is a solution to the wave equation

$$u_{tt} = a^2 u_{xx}$$

for  $t > 0$  and all values of  $x$ .

**Solution 13.7:** To do this, we need to find  $u_{tt}$  and  $u_{xx}$  and show that the equation holds.

$$\begin{aligned}u_t &= a[2(at)] = 2a^2t \\ \implies u_{tt} &= 2a^2\end{aligned}$$

$$\begin{aligned}u_x &= 2x \\ \implies u_{xx} &= 2\end{aligned}$$

Plugging into the wave equation, we get

$$\begin{aligned}[2a^2] &= a^2 [2] \\ \implies 2a^2 &= 2a^2\end{aligned}$$

Since our result is trivially true, then we know  $u = x^2 + (at)^2$  is a solution to the wave equation.

**Example 13.8:3.** Show that  $u = a^2x^2 \ln(t)$  is **not** a solution to the heat equation

$$u_t = a^2u_{xx}$$

for  $t > 0$  and all values of  $x$ .

**Solution 13.9:** To do this, we need to find  $u_t$  and  $u_{xx}$  and show that the equation holds.

$$u_t = \frac{a^2x^2}{t}$$

$$\begin{aligned}u_x &= 2xa^2 \ln(t) \\ \implies u_{xx} &= 2a^2 \ln(t)\end{aligned}$$

Plugging into the heat equation, we get

$$\begin{aligned}\frac{a^2x}{t} &= 2a^2 \ln(t) \\ \frac{x}{t} &= 2 \ln(t)\end{aligned}$$

This statement is FALSE. This will not hold for all values of  $x$  and  $t$ .

### Summary of Ideas: Lecture 13

- We can take a partial derivative more than once. The number of times we take a partial derivative determines the order of the derivative.
- The order in which we take partial derivatives does not matter. That is,  $f_{xyz} = f_{yzx} = f_{zyx} = f_{yxz} = f_{zxy} = f_{xzy}$ .
- We can determine if a function is a solution to a partial differential equation by plugging it into the equation.

PENN STATE UNIVERSITY, UNIVERSITY PARK, PA 16802