

12. LECTURE 12

Objectives

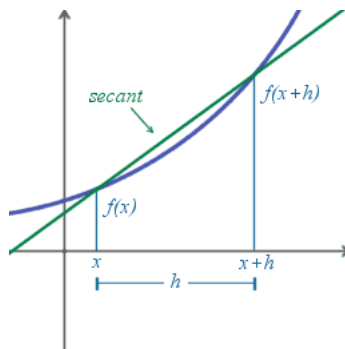
- I can approximate a partial derivative from raw data.
- I understand the notation for partial derivatives.
- I can identify if a partial derivative is increase of decreasing by its image.
- I know how to take a partial derivative with respect to a variable.

At the end of the previous section, we discussed directions relative to a level curve and the extent to which the dependent variable changes. Before we can discuss how to properly define these arrows (which are vectors), we need to understand *how much* the height change relative to each independent variable. This is information we get from a derivative.

When you studied derivatives in Math 110, you learned that they tell us the extent to which the dependent variable changes with respect to the independent variable. The limit definition of a derivative was

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

This limit is constructed by finding the slope of a secant line from x to $x+h$. That is, the numerator comes from the change in y and the denominator comes from the change in x .



In real life, we often don't have a function that is modeling reality. Instead, we are trying to determine a rate of change (a derivative) from raw data. When this is the case, we cannot find the true derivative. Instead, we *approximate* that derivative using a secant line. Let's consider the data set below. It describes the *humidex*, a function that tells you how hot it *feels* given the actual temperature (T) and humidity (H).

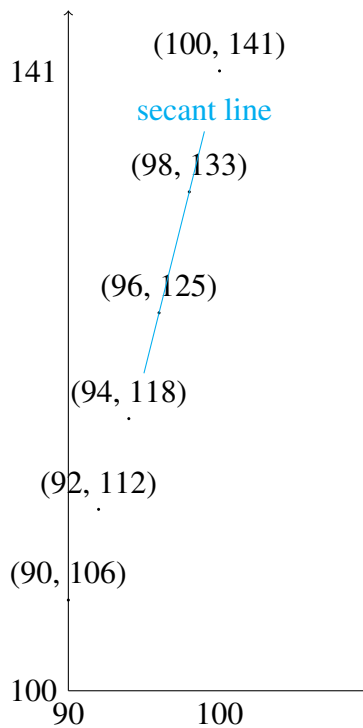
		Relative humidity (%)								
$T \backslash H$		50	55	60	65	70	75	80	85	90
Actual temperature (°F)	90	96	98	100	103	106	109	112	115	119
	92	100	103	105	108	112	115	119	123	128
	94	104	107	111	114	118	122	127	132	137
	96	109	113	116	121	125	130	135	141	146
	98	114	118	123	127	133	138	144	150	157
	100	119	124	129	135	141	147	154	161	168

What is the derivative in higher dimensions?

$$g(T) = f(T, 70)$$

$$g'(96) = \lim_{h \rightarrow 0} \frac{g(96 + h) - g(96)}{h} = \lim_{h \rightarrow 0} \frac{f(96 + h, 70) - f(96, 70)}{h}$$

Suppose we wanted to understand the rate at which temperature increases relative to humidity at $96^\circ F$ and 70% humidity. That means, we want to understand $g'(96)$ where $g(T) = f(T, 70)$. Recall from the previous section that when we plug in a constant for one variable, we are actually considering a two-dimensional function. Hence, we can graph it and understand it. In the case of $g(T)$, we do not know its actual function, but we can plot its data.



We could try to fit the data to a function and find that function's derivative. A way to estimate the derivative directly is to approximating using the secant line that connects $g(96)$ with the next available data point, $g(98)$. That is,

$$g'(96) \approx \frac{g(x+h) - g(x)}{h} = \frac{133 - 125}{2} = 4$$

So the derivative is approximately 4. That means when the temperature is $96^\circ F$ and the humidity is 70%, then the temperature feels like it increases by 4 degrees for every one-degree increase in T , the actual temp.

What did we do? We found derivative with respect to only one independent variable while keeping the other constant. We call this a **partial derivative**.

Definition 12.1: A **partial derivative** is a derivative taken with respect to one independent variable, treating all other independent variables as constants.

To denote the specific derivative, we use subscripts. For example, the derivative of f with respect to x is denoted f_x .

What does it mean when we take a partial derivative? A partial derivative is the slope of a tangent line that only changes with respect to one variable. In the image below, the slope of T_1 is the partial derivative with respect to x at (a, b, c) . This is denoted by $f_x(a, b)$. The slope of T_2 is $f_y(a, b)$.

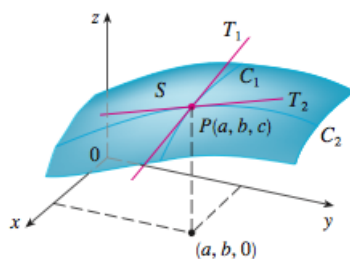


FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2 .

How can we tell if the slope is positive or negative? The arrows of the axes point in the direction that the variable is increase. As y increases, T_2 rises. Therefore, $f_y(a, b) > 0$. As x increases, T_1 falls. Therefore $f_x(a, b) < 0$.

Algebraically, finding a partial derivative means taking a derivative with respect to *only one* variable, and treating all others as constants.

Let's look at some examples of calculating partial derivatives from functions.

For the following examples, the **color blue** will indicate a portion of the function that is treated as a **constant**. Think of these portions as being numbers. The portions that have changed (because of a derivative) are in **red**.

Example 12.2:1. Find the first partial derivatives of the function

$$f(x, y) = \frac{1}{x^2} e^{-y}$$

Since there is only two variables, there are two first partial derivatives. First, let's consider f_x . In this case, y is fixed and we treat it as a constant. So, e^{-y} is just a constant.

$$f_x(x, y) = -\frac{2}{x^3} e^{-y}$$

Now, find f_y . Here, x is fixed so $\frac{1}{x^2}$ is just a constant.

$$f_y(x, y) = -e^{-y} \frac{1}{x^2}$$

Example 12.3:2. Find the first partial derivatives of the function

$$f(x, y) = x^4y^3 + 8x^2y$$

Again, there are only two variables, so there are only two partial derivatives. They are

$$f_x(x, y) = 4x^3y^3 + 16xy$$

and

$$f_y(x, y) = 3x^4y^2 + 8x^2$$

Summary of Ideas: Lecture 12

- Derivatives can be approximated using the limit definition. That is

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

- We can determine if a partial derivative is positive or negative by considering the change in the dependent variable as the relevant independent variable increases.
- A partial derivative with respect to a variable, takes the derivative of the function with respect to that variable and treats all other variables as constants.