Objectives

- ☐ I know how to graph level curves when there are two input variables.
- ☐ I can find the line of steepest descent, steepest ascent, or zero height change.

In this lecture, we will study functions that take in multiple scalar inputs, like x and y, but produce just one scalar output

$$z = f(x, y).$$

These are called *functions of several variables*. They are the main object of study in multivariate calculus.

The first step in understanding any function is being able to graph it. Unfortunately, graphing functions in more than three dimensions quite tricky. One popular method for graphs of three and four dimensions is to graph their *level curves*.

Definition 11.1: A **level curve** of a multivariate function is any function constructed by choosing a specific value for the dependent variable.

Each level curve gives us a height of the multivariate function. For functions of three variables, we can use this information to construct the original function.

Example 11.2: Graph the level curves of

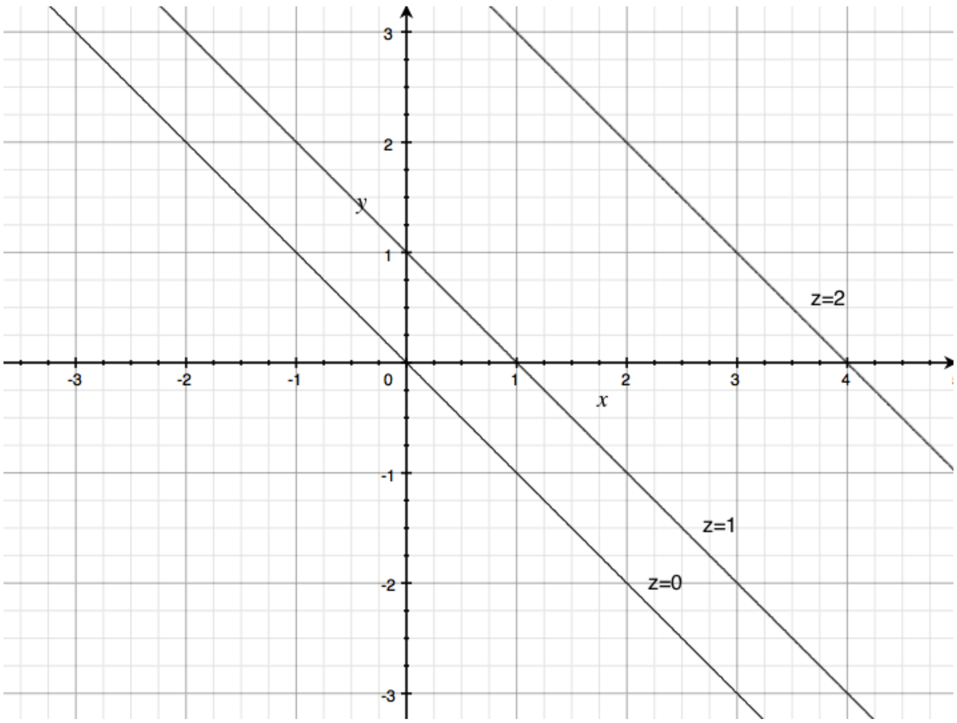
$$f(x, y) = \sqrt{x + y}$$

Use that information to sketch the 3 dimensional graph.

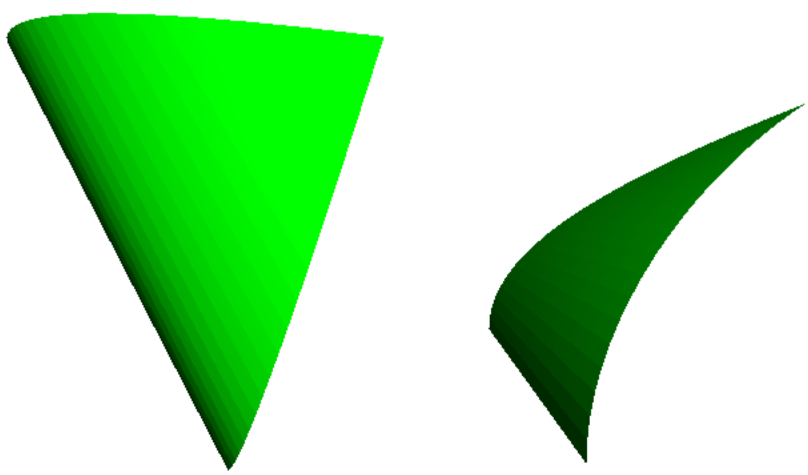
Solution 11.3: Let's pick values for f and write the respective functions.

f	Function
0	$0 = \sqrt{x+y} \implies 0 = x + y$
1	$1 = \sqrt{x+y} \implies 1 = x + y$
2	$2 = \sqrt{x+y} \implies 4 = x + y$
3	$3 = \sqrt{x+y} \implies 9 = x + y$

Now, we graph all these lines on the same graph.



From this, we see that the curve is increasing in height but the increasing is slowing down. Here is the graph of the three dimensional surface seen from two different angles.



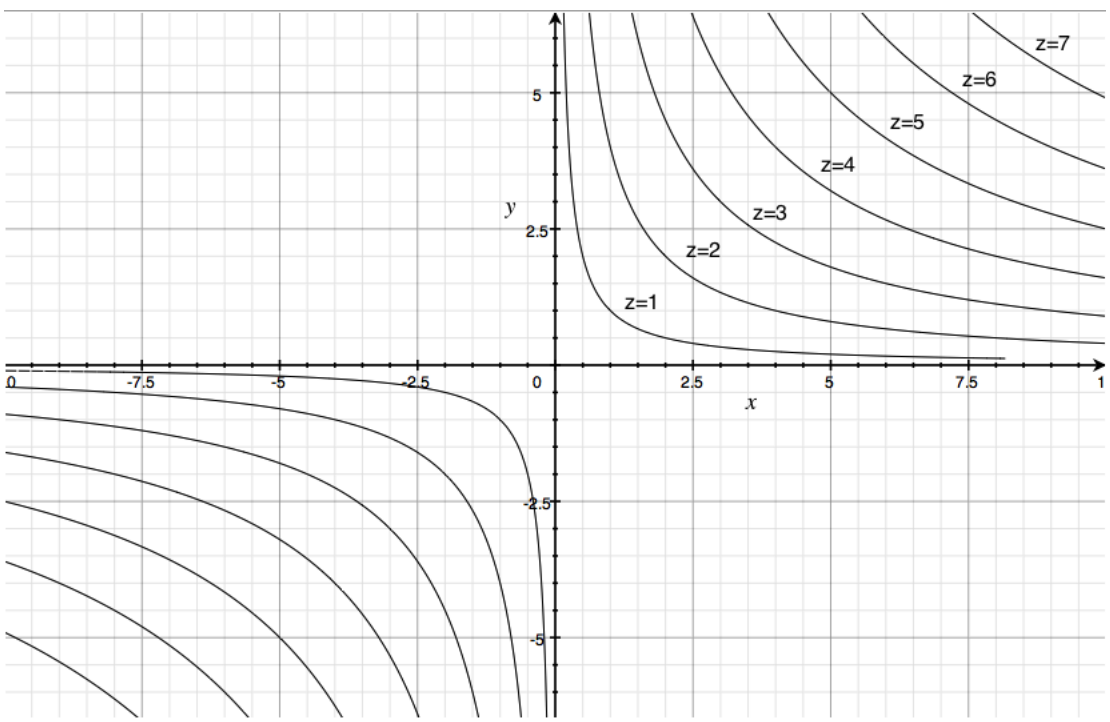
Example 11.4: Graph the level curves of

$$f(x, y) = \sqrt{xy}$$

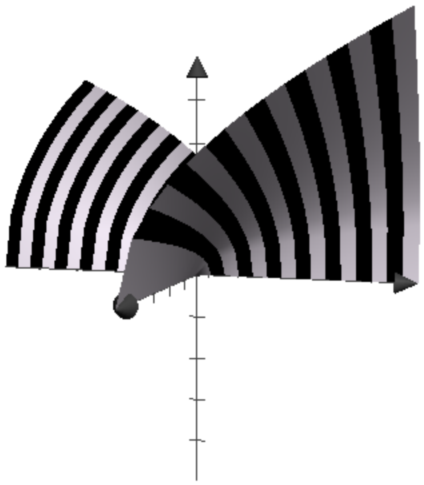
Use that information to sketch the 3 dimensional graph.

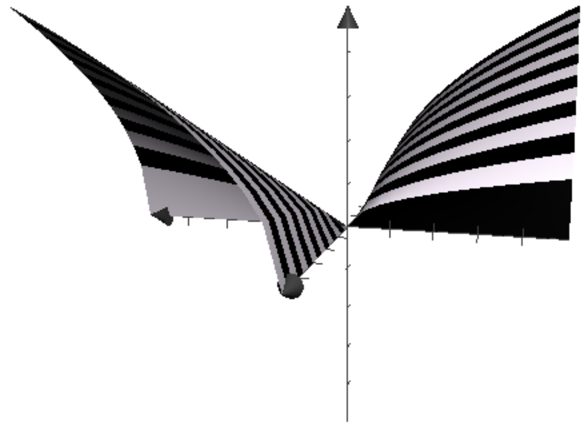
Solution 11.5: First, we pick z values. In the picture below, I have graphed $z = 1$, $z = 2$, ..., and $z = 7$.

f	Function
0	$0 = \sqrt{xy}$
1	$1 = \sqrt{xy}$
2	$2 = \sqrt{xy}$
3	$3 = \sqrt{xy}$
4	$4 = \sqrt{xy}$
5	$5 = \sqrt{xy}$
6	$6 = \sqrt{xy}$



Given the shape of the level curve, our graph is then





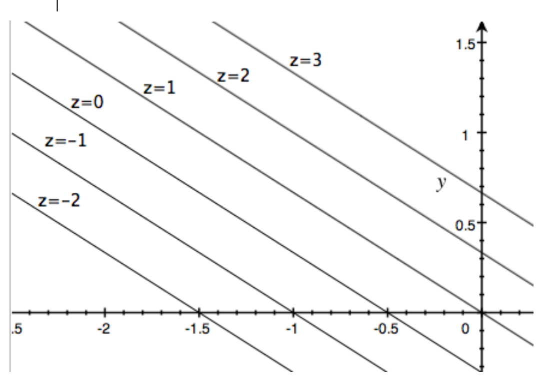
Example 11.6: Graph the level curves of

$$f(x,y) = 2x + 3y + 1$$

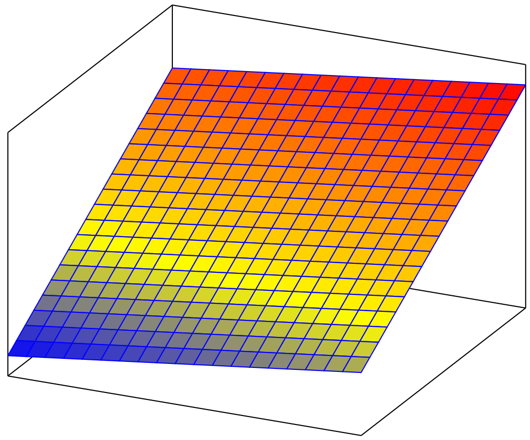
Use that information to sketch the 3 dimensional graph.

Solution 11.7: First, we pick z values.

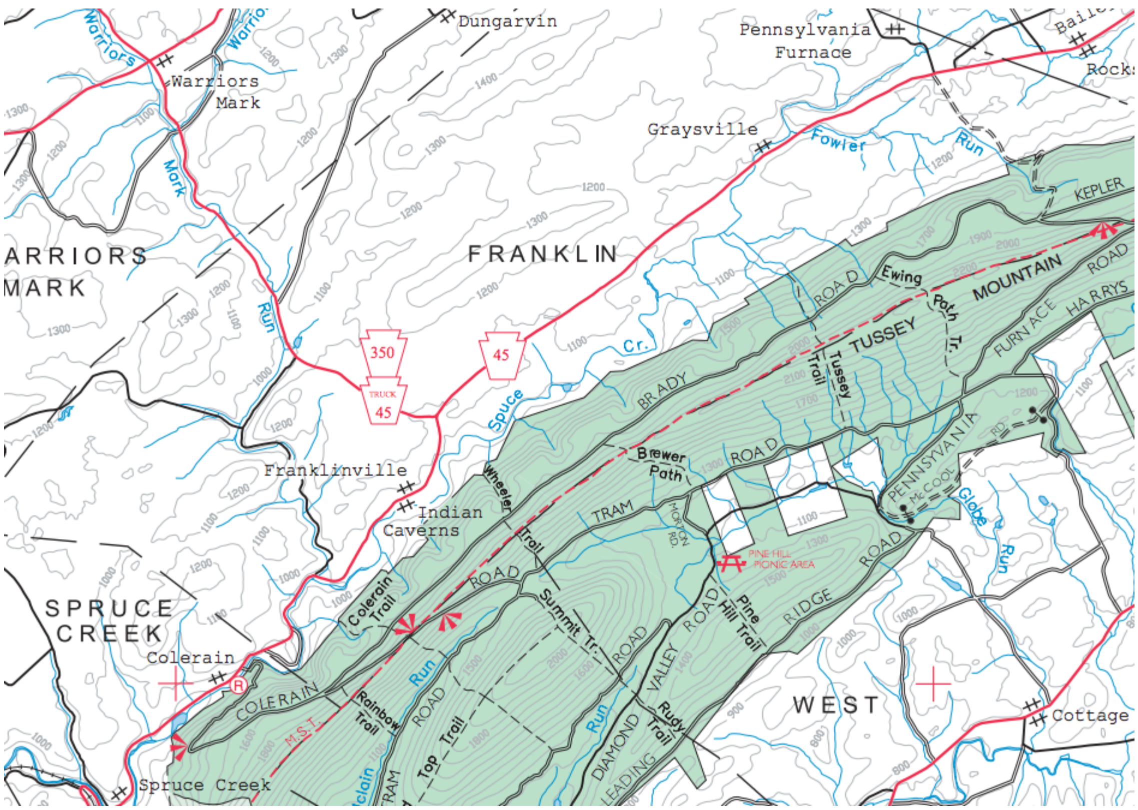
f	Function
-2	$-2 = 2x + 3y + 1 \implies y = -\frac{2x+3}{3}$
-1	$-1 = 2x + 3y + 1 \implies y = -\frac{2x+2}{3}$
0	$0 = 2x + 3y + 1 \implies y = -\frac{2x+1}{3}$
1	$1 = 2x + 3y + 1 \implies y = -\frac{2x}{3}$
2	$2 = 2x + 3y + 1 \implies y = -\frac{2x-1}{3}$
3	$3 = 2x + 3y + 1 \implies y = -\frac{2x-2}{3}$



Given the shape of the level curve, our graph is a plane.



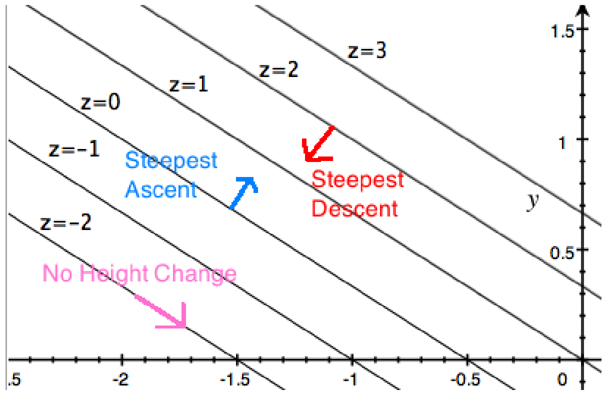
These are the lines one sees on a topographic map. Below is an example of a topographic map of Tussey Mountain.



Notice the lines looping around with jagged edges. Around the word “FRANKLIN,” we see a loop labeled 1300. This tells us that on that line, the elevation is 1300ft.

This technique is a way of understanding the dependent variable as a height. To understand how the dependent variable changes with respect to the independent variables, we will consider the level curves.

Consider the level curves of the plane in the previous example. What vector points toward the steepest ascent? What vector points toward the steepest descent? As you can see from the image below, both directions are perpendicular to the level curves. Therefore, any vector perpendicular to the level curve is pointing in the direction of the largest change. For the same reason, any vector that is parallel to a level curve points toward the direction of no change.



In more general terms, when we consider the level curves of an arbitrary function, the largest changes in the dependent variable are experienced in directions perpendicular to the level curves. No changes in height are experienced along the level curve.

Summary of Ideas: Lecture 11

- The **level curves** of a multivariate function are the lines for various values of the dependent variable f.
- Drawing level curves is a technique for graphing three-dimensional surfaces.
- The directions of steepest ascent and descent are perpendicular to the level curves.
- Directions that are parallel to level curves are where the heights do not change.

PENN STATE UNIVERSITY, UNIVERSITY PARK, PA 16802