The Field of Complex Numbers – Algebraic

Closure of a Non-Archimedean Real Closed Field

By: Sankha Subhra Basu

Essay Advisor: Dr. Wim Ruitenburg

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Department of Mathematics, Statistics, and Computer Science
Marquette University
P.O. Box 1881
Milwaukee, WI 53201-1881

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**The problem**

We know that the field of Complex numbers is an algebraically closed and complete field extension of the field of Rational numbers and the field of Real numbers. The field of Rational numbers is neither algebraically closed nor complete. The field of Real numbers even though complete is not algebraically closed. But both of these are Archimedean fields. In this essay we sketch a proof of the known result that there is a non-Archimedean real closed field whose algebraic closure is the field of Complex numbers. We will use model theoretic techniques. We will also use some well known results from algebra.

**Real closed field:** A field is called formally real if \(-1\) is not expressible in it as a sum of squares. A field \(P\) is called a real closed field if \(P\) is formally real but no proper algebraic extension of \(P\) is formally real.

**Facts about real closed fields:**

(i) A formally real field and hence a real closed field has zero characteristic; for, in a field of characteristic \(p\), \(-1\) is always a sum of \(p-1\) summands \(1^2\).

(ii) Every real closed field can be ordered in one, and only one, way such that, the collection of elements greater than 0 is closed under sum and product.

(iii) In a real closed field every polynomial of odd degree has at least one root.

(iv) In a real closed field every positive element has a square root.

(v) A real closed field is not algebraically closed but the field arising by the adjunction of \(i = \sqrt{-1}\) is algebraically closed.
**Archimedean field:** An ordered field is called Archimedean or, the ordering of a field is called Archimedean if for every field element \( a \), there exists a natural number \( n \), such that \( 1+1+\ldots +1 \) (\( n \) times) \( > a \).

Thus a **non-Archimedean field** is an ordered field where this is not the case. That is, there is a field element \( a \), which is greater than \( 1+1+\ldots +1 \) (\( n \) times) for every natural number \( n \).

We shall now give an axiomatization for non-Archimedean real closed fields over a suitable language and show that there exists such a field, that is, the set of axioms is satisfiable.

The language \( \mathcal{L} \) that we consider here has countably many variables; three constant symbols, ‘0’, ‘1’, ‘c’; one unary function symbol, ‘−’; two binary function symbols, ‘+’, ‘×’; one unary relation symbol ‘\( P \)’; one binary relation symbol, ‘\( = \)’. We also have the logical connectives, ‘\( \neg \)’ ‘\( \land \)’, ‘\( \lor \)’ ‘\( \rightarrow \)’, ‘\( \leftrightarrow \)’ and the quantifiers, ‘\( \forall \)’, ‘\( \exists \)’.

We define for each natural number \( n \geq 1 \), \( n := 1+1+\ldots +1 \) (\( n \) times). Also, we shall write \( x\times y \) for \( x \times y \), \( x + y \) for \( x + y \). For any term \( t \) of the language, and for any natural number \( p \geq 1 \), \( t^p := t \times t \times \ldots \times t \) (\( p \) times).

We consider the following set of formulas \([2, 4]\Phi \), as our proper axioms apart from the usual first-order logical axioms.

(a) **Field axioms:**

\[
\forall x \forall y \forall z (x + (y + z) = (x + y) + z);
\]

\[
\forall x \forall y (x + y = y + x);
\]
∀x (x + 0 = x);
∀x (x + (−x) = 0);
∀x∀y∀z (x (yz) = (xy) z);
∀x∀y (xy = yx);
∀x (x1 = x);
∀x∃y (x = 0 ∨ xy = 1);
∀x∀y∀z (x (y+z) = xy + xz);
¬ (0 = 1).

(b) **Positivity axioms:**

∀x∀y (Px ∧ Py → P(x + y));
∀x (x = 0 ∨ Px ∨ P(−x));
∀x ¬ (Px ∧ P(−x));
∀x∀y (Px ∧ Py → P(xy));

(c) **Additional axioms for real closed fields:**

∀x∃y (x = y² ∨ −x = y²);
∀x₀∀x₁…∀x₂n∃x (x₀ + x₁x +…+ x₂n x²ⁿ + x²ⁿ⁺¹ = 0) for each natural number n ≥ 1.

We introduce another binary relation symbol ‘>’ and define it by >xy := P(x−y). We shall write x > y for >xy. We now show that ‘>’ defines a linear ordering.

Suppose, x > y and y > z. Then P(x−y) and P(y−z). Thus, P((x−y) + (y−z)) that is, P(x−z) and so x > z. Thus, we have transitivity: ∀x∀y∀z (x > y ∧ y > z → x > z).
For any \( x \) and \( y \), either \( x - y = 0 \) or, \( P(x - y) \) or, \( P(-(x - y)) \) that is, either \( x = y \) or, \( x > y \) or, \( y > x \). Thus, we have trichotomy: \( \forall x \forall y \ (x = y \lor x > y \lor y > x) \).

From the field axioms, we can deduce that \(-0 = 0\). Thus, in lieu of the order axiom, \( \forall x \neg (Px \land P(-x)) \), we conclude that \( \neg P0 \). Then using the field axiom, \( \forall x \ (x + (-x) = 0) \) we get, \( \forall x \neg (P(x - x)) \). Thus, we have irreflexivity: \( \forall x \neg (x > x) \).

For any \( x, y, z, w \) such that \( x > y \) and \( z > w \), that is, \( P(x - y) \) and \( P(z - w) \), we have \( P((x - y) + (z - w)) \). Thus, \( P((x + z) - (y + w)) \) and hence, \( x + z > y + w \). So, \( \forall x \forall y \forall z \forall w \ (x > y \land z > w \rightarrow x + y > z + w) \).

For any \( x, y, z \) such that \( x > 0 \) and \( y > z \), that is, \( Px \) and \( P(y - z) \), we have \( P(x(y - z)) \). Now, from the field axioms, \( x(y - z) = xy - xz \). Thus, \( \forall x \forall y \forall z \ (x > 0 \land y > z \rightarrow xy > xz) \).

Hence, ‘\( > \)’ defines a linear ordering which respects sum and product.

For any \( x \), either \( x = 0 \) or, \( Px \) or, \( P(-x) \). If \( x = 0 \), then from the field axioms we can deduce that \( x^2 = 0 \). If \( Px \), then from the order axioms, \( P(x^2) \). Lastly, if \( P(-x) \), then \( P((-x)^2) \). It is derivable from the field axioms that \( (-x)^2 = x^2 \). Thus, we get that in an ordered field, \( \forall x \ (x = 0 \lor P(x^2)) \). Thus, by using the order axiom \( \forall x \forall y \ (Px \land Py \rightarrow P(x + y)) \) finitely many times, we get \( \forall x_0 \forall x_1 \ldots \forall x_n \ ((x_0 = 0 \land x_1 = 0 \land \ldots \land x_n = 0) \lor P(x_0^2 + x_1^2 + \ldots + x_n^2)) \) for each natural number \( n \geq 0 \). Now, we note that \( \neg (-1 = 0) \) and \( \neg P(-1) \). Thus, we get that in an ordered field and hence in a real closed field, \(-1\) cannot be expressed as a sum of squares, that is, \( \forall x_0 \forall x_1 \ldots \forall x_n \neg ((x_0^2 + x_1^2 + \ldots + x_n^2 = -1)) \) for each natural number \( n \geq 0 \).

We now define the following formulas:
\( \varphi_0 := c > 0; \varphi_1 := c > 1; \) and in general, \( \varphi_n := c > n \) for any natural number \( n \geq 0 \).

Let \( \Psi := \Phi \cup \{ \varphi_n : n \geq 0 \} \).

Then \( \Psi \) is an axiomatization for non-Archimedean real closed fields. We now need to show that \( \Psi \) is satisfiable, that is, \( \Psi \) has a model.

**Compactness Theorem:**[3] A set of formulas \( \Gamma \) is satisfiable if and only if every finite \( \Gamma_0 \subseteq \Gamma \) is satisfiable.

So with the compactness theorem at our hand, to show that \( \Psi \) is satisfiable, it suffices to show that every finite subset of \( \Psi \) is satisfiable.

We note that any finite subset of \( \Psi \) is the union of a finite subset of \( \Phi \) and a finite subset of the set \( \{ \varphi_n : n \geq 0 \} \). Let \( \Psi' \) be a finite subset of \( \Psi \).

Then \( \Psi' \subseteq \Phi \cup \{ \varphi_n : n > k \} \) for some \( n > 0 \).

Let \( \Psi'' = \Phi \cup \{ \varphi_n : n > k \} \). To show that \( \Psi' \) has a model, it suffices to show that \( \Psi'' \) has a model because every model of \( \Psi'' \) is also a model of \( \Psi' \).

Let \( \mathbb{R}^> \) be the ordered field of real numbers. Then \( \mathbb{R}^> \) is a real closed field and hence, satisfies \( \Phi \). Let \( M \) be a sufficiently large real number, that is, \( M > n \). Then \( \mathbb{R}^> \) along with \( M, (\mathbb{R}^>, M) \) satisfies \( \Psi'' \). Thus, \( \Psi' \) has a model. Now, since \( \Psi' \) was an arbitrarily chosen finite subset of \( \Psi \), we conclude that every finite subset of \( \Psi \) has a model. Hence, by the Compactness Theorem, there is a non-Archimedean real closed field.
Moreover, since every real closed field is of characteristic zero, every non-Archimedean real closed field is of characteristic zero. Every field of characteristic zero has an infinite domain. So, every non-Archimedean real closed field is infinite and has characteristic zero. In other words, $\Psi$ is satisfiable only over an infinite domain.

**Theorem of Löwenheim, Skolem and Tarski:** Let $\Lambda$ be a set of formulas which is satisfiable over an infinite domain and let $\kappa$ be an infinite cardinal greater than or equal to the cardinality of $\Lambda$. Then $\Lambda$ has a model of cardinality $\kappa$.

By the construction of $\Psi$, we note that $\Psi$ is countable. Thus, using the Theorem of Löwenheim, Skolem and Tarski, we conclude that $\Psi$ has a model whose domain is of cardinality continuum. Let $F$ be a non-Archimedean real closed field with domain of cardinality continuum. Since $F$ is a real closed field, $F(i)$ is algebraically closed and with domain of cardinality continuum.

Now, since $F$ is of characteristic zero, its prime subfield is isomorphic to the field of rational numbers, $Q$. Let $\Omega$ be the field of algebraic numbers. $\Omega$ is the smallest algebraically closed field containing $Q$. Thus, the algebraic closure of $F$, $F(i)$ contains an isomorphic copy of the field of algebraic numbers, $\Omega$. Since $\Omega$ is of cardinality $\omega$ (cardinality of the natural numbers), $F(i)$ properly contains $\Omega$ (up to isomorphism) and moreover, the transcendence degree of $F(i)$ over $\Omega$ must be continuum.

We know that the field of complex numbers, $C$, is an algebraically closed extension of $\Omega$ with transcendence degree continuum.
Thus, $F(i)$ and $C$ are both algebraically closed extensions of the algebraically closed field $\Omega$, with transcendence bases of the same cardinality continuum. Hence, by a well-known result from algebra, $F(i)$ and $C$ are isomorphic.$^5$

So, we conclude that the field of complex numbers $C$ is the algebraic closure of a non-Archimedean real closed field.
References


