Solution of Linear Systems by the Echelon Method

A **first-degree equation in n unknowns** is any equation of the form

\[ a_1x_1 + a_2x_2 + \cdots + a_nx_n = k, \]

where \( a_1, a_2, \ldots, a_n \) and \( k \) are real numbers and \( x_1, x_2, \ldots, x_n \) represent variables.

A **solution** of the first degree equation

\[ a_1x_1 + a_2x_2 + \cdots + a_nx_n = k \]

is a sequence of numbers \( s_1, s_2, \ldots, s_n \) such that

\[ a_1s_1 + a_2s_2 + \cdots + a_n s_n = k, \]

and is written as \((s_1, s_2, \ldots, s_n)\).

When there are only two variables involved, first-degree equations are just our good old linear equations, because their graphs are straight lines. Because of this, we refer to all first degree equations as linear equations.

A **system of linear equations** is a collection of at least two linear equations. A **solution** to a system of linear equations, or a linear system is a sequence of numbers that simultaneously satisfy all the equations in the system.

The following are examples of linear systems:

\[
\begin{align*}
2x + 3y &= 12 \\
3x - 4y &= 1.
\end{align*}
\]

\[
\begin{align*}
2x + y - z &= 2 \\
x + 3y + 2z &= 1 \\
x + y + z &= 2.
\end{align*}
\]

To solve a linear system of equations, we use properties of algebra to change, or transform, the system into a simpler **equivalent** system. An **equivalent system** is one that has the same solutions as the given system.

**What can you do?** - **Transformations of a system:**

- Exchange any two equations;
- Multiply both sides of an equation by a non-zero number;
• Replace any equation by a non-zero multiple of that equation plus a non-zero multiple of any other equation of the system.

The Echelon Method of Solving a Linear System

1. If possible, arrange the equations so that there is an \( x_1 \)-term in the first equation, an \( x_2 \)-term in the second equation, and so on.

2. Eliminate the \( x_1 \)-term in all equations after the first equation. Use the first equation for this.

3. Eliminate the \( x_2 \)-term in all equations after the second equation. Use the second equation for this.

4. Eliminate the \( x_3 \)-term in all equations after the third equation. Use the third equation for this.

5. Continue in this way until the last equation has the form \( ax_n = k \), for constants \( a \neq 0 \) and \( k \), if possible.

6. Multiply each equation by the reciprocal of the coefficient of its first term.

7. Use back-substitution to find the value of each variable.

If the variables involved are \( x, y, z \), treat \( x \) as \( x_1 \), \( y \) as \( x_2 \), and \( z \) as \( x_3 \).

Types of Solutions

• Exactly one solution: In this case, we say that the system is \textit{consistent and independent}.

• Infinitely many solutions: In this case, we say that the system is \textit{consistent and dependent}. The solutions are expressed in terms of a \textit{parameter}. The right-most variable is commonly chosen as the parameter.

• No solution: In this case, we say that the system is \textit{inconsistent}.

For examples, see Exam 1 and Homework 4.
Solution of Linear Systems by the Gauss-Jordan Method

A rectangular array of numbers enclosed by brackets is called a *matrix* (plural: *matrices*). Each number in the array is called an *element* or *entry* of the matrix.

A system of equations can be written in an abbreviated form using a matrix. This is called the *augmented matrix*. For example, the augmented matrix for the system of equations

\[
\begin{align*}
2x + y - z &= 2 \\
x + 3y + 2z &= 1 \\
x + y + z &= 2
\end{align*}
\]

is

\[
\begin{bmatrix}
2 & 1 & -1 & | & 2 \\
1 & 3 & 2 & | & 1 \\
1 & 1 & 1 & | & 2
\end{bmatrix}
\]

The vertical line in the augmented matrix separates the constants from the coefficients of the variables.

The rows of the augmented matrix can be transformed in the same way as the equations of the system, since the matrix is just a shortened form of the system. These transformations are called *row operations*.

**What can you do? - Row Operations:**

- Interchange any two rows.
- Multiply each element of a row by a non-zero number.
- Add a non-zero multiple of the elements of one row to the corresponding elements of a non-zero multiple of some other row.

These row operations produce augmented matrices of equivalent linear systems.

The Gauss-Jordan Method of Solving a Linear System

1. Write each equation so that variable terms are in the same order on the left side of the equals sign and constants are on the right. This is the proper form of the linear system.

2. Write the augmented matrix that corresponds to the system.

3. Use row operations to transform the first column so that all elements except the element in the first row are zero. Use row-1 to change all other rows.

4. Use row operations to transform the second column so that all elements except the element in the second row are zero. Use row-2 to change all other rows.

5. Use row operations to transform the third column so that all elements except the element in the third row are zero. Use row-3 to change all other rows.
6. Continue in this way, when possible, until the last row is written in the form

\[
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & j & k
\end{bmatrix},
\]

where \( j \) and \( k \) are constants with \( j \neq 0 \). When this is not possible, continue until every row has more zeros on the left than the previous row (except possibly for any row of all zeros at the bottom of the matrix), and the first non-zero entry in each row is the only non-zero entry in its column.

7. Multiply each row by the reciprocal of the first non-zero element in that row.

For a system of 3 equations in 3 variables, the following are the possible final forms of the augmented matrices:

- **Inconsistent system:**
  \[
  \begin{bmatrix}
  1 & 0 & * \\
  0 & 1 & * \\
  0 & 0 & 0
  \end{bmatrix}
  \]

  [\( A * \) stands for some number (zero or non-zero), whereas a \( ** \) stands for some non-zero number.]

  There is no solution.

- **Consistent and Dependent system:**
  \[
  \begin{bmatrix}
  1 & 0 & * \\
  0 & 1 & * \\
  0 & 0 & 0
  \end{bmatrix}
  \]
  or
  \[
  \begin{bmatrix}
  1 & * & * \\
  0 & 0 & 0 \\
  0 & 0 & 0
  \end{bmatrix}
  \]
  [\( A * \) stands for some number (zero or non-zero).]

  There are infinitely many solutions. The general solution will be in terms of one parameter for a system with a matrix of the first form, and in terms of two parameters for a system with a matrix of the second form.

- **Consistent and Independent system:**
  \[
  \begin{bmatrix}
  1 & 0 & * \\
  0 & 1 & * \\
  0 & 0 & 1
  \end{bmatrix}
  \]

  [\( A * \) stands for some number (zero or non-zero).]

  There is exactly one solution to this system.

For examples, see Quiz 2, Exam 2 and Homework 5.
Addition and Subtraction of Matrices

Size of a Matrix: Matrices are classified by size; that is, by the number of rows and columns they contain. A matrix with $m$ rows and $n$ columns is called an $m \times n$ matrix.

- A matrix with the same number of rows as columns is called a square matrix.
- A matrix containing only one row is called a row matrix or row vector.
- A matrix containing only one column is called a column matrix or column vector.

Matrix Equality: Two matrices are equal if they are the same size and if their corresponding elements are equal.

Adding Matrices: The sum of two $m \times n$ matrices $A$ and $B$ is the $m \times n$ matrix $A + B$ in which each element is the sum of the corresponding elements of $A$ and $B$.

Subtracting Matrices: The difference of two $m \times n$ matrices $A$ and $B$ is the $m \times n$ matrix $A - B$ in which each element is the difference of the corresponding elements of $A$ and $B$.

[ADDITION AND SUBTRACTION CAN BE DONE ONLY ON MATRICES THAT ARE THE SAME SIZE.]

Zero Matrix: A matrix whose elements are all zeros is called a zero matrix.

There is an $m \times n$ zero matrix for each pair of values of $m$ and $n$. Such a matrix serves as an $m \times n$ additive identity, that is, for any $m \times n$ matrix, $A$, if $O$ is the $m \times n$ zero matrix, then

$$A + O = O + A = A.$$  

Multiplication of Matrices

In work with matrices, a real number is called a scalar.

Product of a Matrix and a Scalar: The product of a scalar $k$ and a matrix $A$ is the matrix $kA$, each of whose elements is $k$ times the corresponding element of $A$.

Product of two Matrices: Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times k$ matrix. To find the element in the $i$th row and $j$th column of the product matrix $AB$, multiply each element in the $i$th row of $A$ by the corresponding element in the $j$th column of $B$, and then add these products. The product matrix $AB$ is an $m \times k$ matrix.

- The product $AB$ of two matrices $A$ and $B$ can be found only if the number of columns of $A$ is the same as the number of rows of $B$. 
• The order of multiplication of matrices is important. The product of two matrices may exist if they are multiplied in one order, but not in the reverse order. For two matrices, $A$ and $B$, even if $AB$ and $BA$ both exist, they may be unequal.

However, we have the following properties of matrix multiplication:

• For appropriate matrices, $A, B, C$, $A(BC) = (AB)C$.

• For appropriate matrices, $A, B, C$, $A(B + C) = AB + AC$.

For examples, see Quiz 3, Exam 2 and Homework 6.
Matrix Inverses

Identity Matrix: An identity matrix is a square matrix, $I$ such that if $A$ is any other square matrix of the same size, then

\[ AI = IA = A. \]

The $2 \times 2$ identity matrix is
\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]
the $3 \times 3$ identity matrix is
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]
and so on.

Inverse of a Matrix: For a square matrix, $A$, if there exists another square matrix, $B$ of the same size such that

\[ AB = BA = I, \]

where $I$ is the identity matrix of the same size as $A$, then we say that the matrix $A$ is invertible, and $B$ is called the inverse of $A$. We then denote this inverse matrix by $A^{-1}$.

- Non-square matrices do not have inverses.
- Not all square matrices have inverses.
- For a square matrix, $A$, if $A^{-1}$ exists, then

\[ AA^{-1} = A^{-1}A = I, \]

the identity matrix of the same size as $A$.

Finding the Inverse of an $n \times n$ Matrix, $A$

1. Form the augmented matrix $[A \mid I]$, where $I$ is the $n \times n$ identity matrix.
2. Perform row operations on $[A \mid I]$ to get a matrix of the form $[I \mid B]$, if this is possible.
3. Matrix $B$ is $A^{-1}$.

Solving a System of Equations with Inverses: A linear system of equations can be written as a matrix equation $AX = B$, where $A$ is the matrix of the coefficients of the variables, $X$ is the matrix of the variables, and $B$ is the matrix of the constants. Matrix $A$ is called the coefficient matrix.

Solving a system $AX = B$ using Matrix Inverses: To solve a linear system of equations $AX = B$, where $A$ is the matrix of coefficients, $X$ is the matrix of the variables, and $B$ is the matrix of the constants, first find $A^{-1}$. Then

\[ X = A^{-1}B. \]

For examples, see Quiz 3 and Exam 2.
Other things to Read:

- More on Matrices (Separate notes)
  - Transpose of a matrix
  - Symmetric matrices
  - Antisymmetric matrices
  - Diagonal matrices
  - Triangular matrices

  For examples, see Quiz 4.

- Determinants (Separate notes)

  For examples, see Quiz 5.

- Input-Output Models (Separate notes)

  For examples, see Homework 7.