Geometriae Dedicata **80:** 201–209, 2000.

Going in Circles: Variations on the Money-Coutts Theorem

SERGE TABACHNIKOV
Department of Mathematics, University of Arkansas, Fayetteville, AR 72701, U.S.A.
e-mail: serge@comp.uark.edu

(Received: 5 April 1999)

Communicated by M. Berger.

Abstract. Given a polygon $A_1, \dots, A_n$, consider the chain of circles: S_1 inscribed in the angle A_1, S_2 inscribed in the angle A_2 and tangent to S_1, S_3 inscribed in the angle A_3 and tangent to S_2, etc. We describe a class of n-gons for which this process is $2n$-periodic. We extend the result to the case when the sides of a polygon are arcs of circles. The case of triangles is known as the Money-Coutts theorem.

Mathematics Subject Classifications (2000): 51M04, 14N99.

Key words: circle geometry, elliptic curves.

1. Introduction

Given a triangle A_1, A_2, A_3, consider the following chain of circles: S_1 is inscribed in the angle A_1; S_2 inscribed in the angle A_2 and tangent to S_1; S_3 inscribed in the angle A_3 and tangent to S_2; S_4 inscribed in the angle A_1 and tangent to S_3; etc. (each circle S_i is the smaller of the two circles tangent to S_{i-1}). It is a lesser known elementary geometry gem that this process is 6-periodic: $S_7 = S_1$ – see Figure 1. Moreover, the 6-periodicity persists if the sides of the triangle are replaced by arcs of circles. This result, known as the Money-Coutts theorem, was published as late as in 1971 – see [1, 3–4] for precise formulations and proofs.

One wonders whether the Money-Coutts theorem extends to other polygons. For example, if P is a regular n-gon (whose vertices are numbered cyclically) then, by symmetry, the circles S_{i-1} and S_{i+1} are congruent for all $i = 1, \dots, n$. It follows that for a regular n-gon the process is $2n$-periodic if n is odd and n-periodic if n is even – see Figure 2.

The next, Mathematica generated, Figure 3 shows that this periodicity is destroyed by a generic perturbation of a regular polygon.

However the periodicity persists for a certain class of polygons. Let $A_1, \dots, A_n$ be the vertices of a convex n-gon P, let $2\alpha_i$ be the interior angle at A_i, and let $a_i = |A_i A_{i+1}|$. As before, inscribe a circle S_1 in the angle A_1, then inscribe a circle S_2 in the angle A_2, tangent to S_1, and continue cyclically. Let O_i be the center of

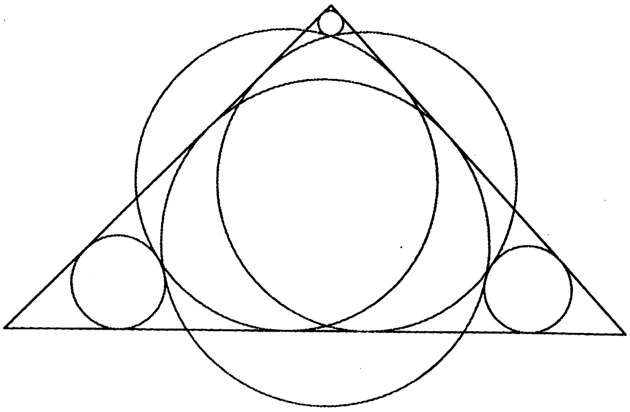

Figure 1.

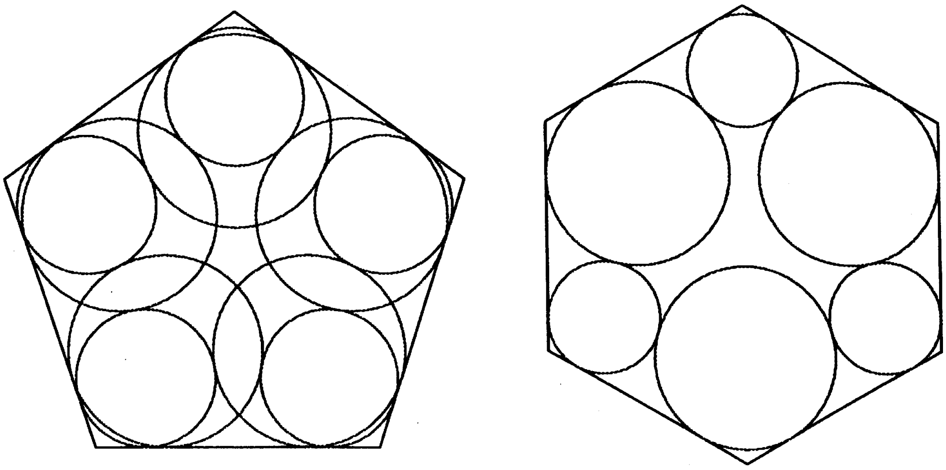

Figure 2.

S_i and r_i its radius. To fix the choice of the circles at each step, assume that the orthogonal projections B_i and B_{i+1} of O_i and O_{i+1} on the line $A_i A_{i+1}$ fall on the segment $A_i A_{i+1}$, and that B_i is closer to A_i than B_{i+1} – see Figure 4.

The Money-Coutts theorem extends to the following class of n-gons P that, for the lack of a better name, we call *nice*. Assume that $n \geqslant 5$, and $\alpha_i + \alpha_{i+1} > \pi/2$ for all i. Let D_i be the intersection point of the lines $A_{i-1}A_i$ and $A_{i+1}A_{i+2}$. Consider the excircles of the triangles $A_{i-1}A_i D_{i-1}$ and $A_i A_{i+1} D_i$ that are tangent to the sides $A_i D_{i-1}$ and $A_i D_i$, respectively. The polygon P is nice if these two circles coincide for all i – see Figure 5.

THEOREM 1. *Let P be a nice n-gon.*

(a) *If n is odd then the sequence of the circles S_i is $2n$-periodic: $S_1 = S_{2n+1}$.*

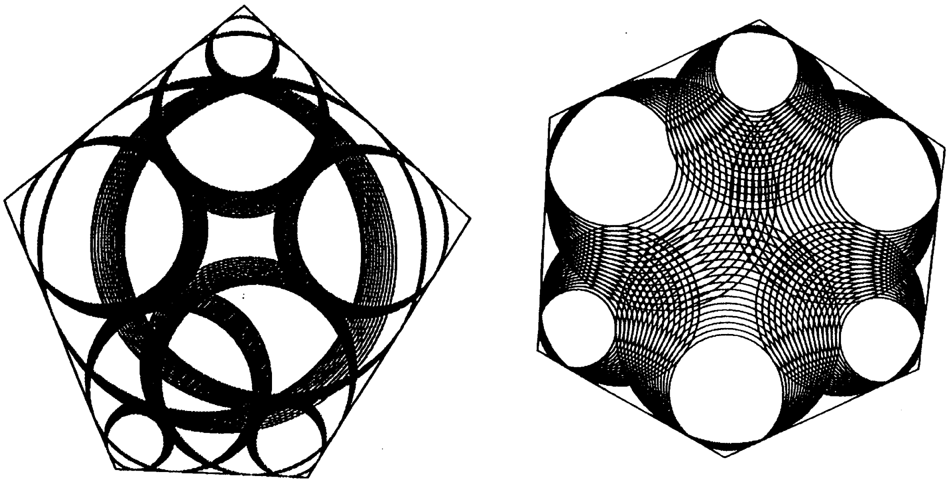

Figure 3.

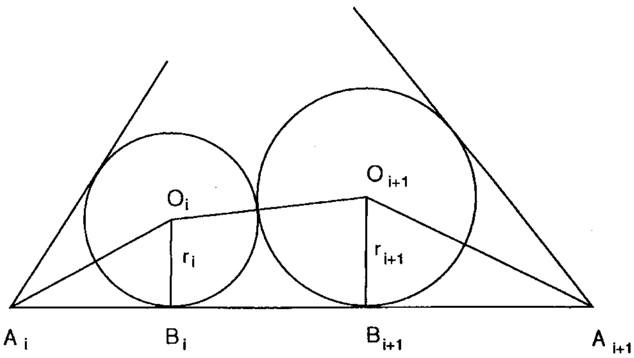

Figure 4.

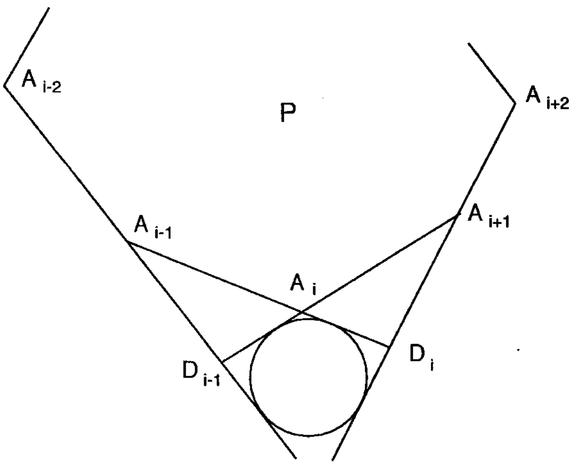

Figure 5.

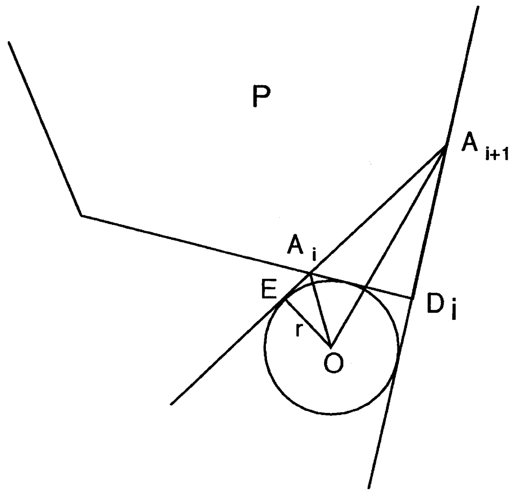

Figure 6.

(b) *If n is even, assume that*

$$\prod_{i=1}^{n}(\sqrt{1-\cot\alpha_i\cot\alpha_{i+1}}+1)^{(-1)^i}=1; \tag{1}$$

then the sequence of the circles S_i is n-periodic: $S_1 = S_{n+1}$.

Proof. First, we claim that P is nice if and only if there exists a constant $\rho > 0$ such that

$$a_i = \rho^2(\tan\alpha_i \tan\alpha_{i+1} - 1) \tag{2}$$

for all i (notice that $\tan\alpha_i \tan\alpha_{i+1} > 1$ since $\alpha_i + \alpha_{i+1} > \pi/2$).

Consider the excircle of the triangle $A_i A_{i+1} D_i$ in Figure 6; let r be its radius. From the right triangle OEA_i one has: $|EA_i| = r\cot\alpha_i$. Likewise, from the right triangle OEA_{i+1} one finds: $|EA_{i+1}| = r\tan\alpha_{i+1}$. It follows that $r\tan\alpha_{i+1} = r\cot\alpha_i + a_i$, and therefore

$$\frac{a_i}{\tan\alpha_i \tan\alpha_{i+1} - 1} = \frac{r}{\tan\alpha_i}. \tag{3}$$

Similarly, let r' be the radius of the excircle of the triangle $A_{i-1}A_iD_{i-1}$; then

$$\frac{a_{i-1}}{\tan\alpha_{i-1}\tan\alpha_i - 1} = \frac{r'}{\tan\alpha_i}. \tag{4}$$

The polygon is nice if and only if $r = r'$; combining (3) and (4), this is equivalent to $a_i/(\tan\alpha_i \tan\alpha_{i+1} - 1)$ being independent of i, as claimed.

Consider Figure 4 again. One has

$$B_iB_{i+1} = 2\sqrt{r_ir_{i+1}}, \qquad |A_iB_i| = r_i \cot\alpha_i,$$
$$|A_{i+1}B_{i+1}| = r_{i+1}\cot\alpha_{i+1},$$

therefore

$$r_i \cot\alpha_i + 2\sqrt{r_ir_{i+1}} + r_{i+1}\cot\alpha_{i+1} = a_i. \tag{5}$$

Introduce new variables

$$u_i = \sqrt{r_i \cot\alpha_i}, \qquad e_i = \sqrt{\tan\alpha_i \tan\alpha_{i+1}}, \qquad c_i = \sqrt{a_i};$$

note that $e_i > 1$. The Equation (5) reads

$$u_i^2 + 2e_iu_iu_{i+1} + u_{i+1}^2 = c_i^2. \tag{6}$$

This equation can be explicitly solved in hyperbolic trigonometric functions. Namely, from (6) one finds

$$u_i + e_iu_{i+1} = \sqrt{c_i^2 + (e_i^2-1)u_{i+1}^2}, \qquad u_{i+1} + e_iu_i = \sqrt{c_i^2 + (e_i^2-1)u_i^2},$$

hence (6) can be rewritten as

$$u_i\sqrt{c_i^2 + (e_i^2-1)u_{i+1}^2} + u_{i+1}\sqrt{c_i^2 + (e_i^2-1)u_i^2} = c_i^2,$$

or, taking (2) into account, as

$$\frac{u_i}{\rho}\sqrt{1+\frac{u_{i+1}^2}{\rho^2}} + \frac{u_{i+1}}{\rho}\sqrt{1+\frac{u_i^2}{\rho^2}} = \frac{c_i}{\rho}. \tag{7}$$

Let $x_i = \sinh^{-1}(u_i/\rho)$ where $\sinh^{-1}(y) = \ln(y+\sqrt{1+y^2})$. Then (7) reads

$$\sinh x_i \cosh x_{i+1} + \sinh x_{i+1}\cosh x_i$$
$$= c_i/\rho \quad \text{or} \quad \sinh(x_i + x_{i+1}) = c_i/\rho. \tag{8}$$

Denote the family of circles inscribed in the ith angle of the polygon P by $\mathcal{F}_i$. We have a map $T_i\colon \mathcal{F}_i \to \mathcal{F}_{i+1}$ that takes S_i to S_{i+1}. One may use x_i as a coordinate in $\mathcal{F}_i$; then (8) means that T_i is a reflection

$$T_i(x_i) = x_{i+1} = \sinh^{-1}(c_i/\rho) - x_i.$$

This implies the statement of the theorem. If n is odd then the return map $T_n T_{n-1} \cdots T_1 \colon \mathcal{F}_1 \to \mathcal{F}_1$ is a reflection, and its second iteration is the identity, as claimed. If n is even then the return map is the translation

$$x_1 \to x_1 + \sum (-1)^i \sinh^{-1}(c_i/\rho). \tag{9}$$

In view of (2), the vanishing of the alternating sum in (9) is equivalent to (1), and we are done.

Remarks. (1) Eliminating ρ from equalities (2), one sees that the codimension of the subvariety of nice n-gons in the $2n$-dimensional space of n-gons is $n - 1$.

(2) If $|e_i| < 1$ then Equation (6) can be similarly solved in trigonometric functions. Moreover, Equation (6) with $|e_i| < 1$ can be considered as the Cosine rule for the triangle with sides u_i, u_{i+1}, c_i. This triangle is inscribed into a circle whose diameter $c_i/\sqrt{1 - e_i^2}$ does not depend on i – see [2]. Likewise, Equation (6) with $e_i > 1$ can be considered as Cosine rule in the Lorentz plane with the metric $\mathrm{d}y^2 - \mathrm{d}x^2$; the respective triangle is inscribed into a pseudo-circle, that is, a hyperbola $x^2 - y^2 = c^2$ with the 'radius' c independent of i.

(3) An intermediate case of $e_i = 1$ in Equation (6) corresponds to the case when the $(i - 1)$st and $(i + 1)$st sides of the polygon are parallel. Then (6) implies that $u_i + u_{i+1} = c_i$. In particular, it follows that if P is a parallelogram then the sequence of circles S_i is not periodic, unless P is a rhombus (in which case $S_5 = S_1$).

(4) It would be interesting to interpret equality (1) geometrically; I do not know such an interpretation.

Next, we discuss how to extend Theorem 1 to the case when the sides of the polygon P are made of arcs of circles. We closely follow [1], Sections 6 and 9, and some of the arguments below are only outlined.

One starts with complexification and projectivization: a circle is considered as a conic $a(x^2 + y^2) + 2bxz + 2cyz + \mathrm{d}z^2 = 0$ in the complex projective plane; the space of circles $\{(a : b : c : d)\}$ is the complex projective space. Given two generic circles C_1 and C_2, the family of circles, tangent to both of them, consists of two irreducible components; each component is a rational curve. The two components are seen in the real situation as follows: if a circle S is tangent to C_1 and C_2 then the line through the two tangency points passes through one of the two centers of similitude of C_1 and C_2; a choice of a component is determined by a choice of one of these two centers.

Let C_1, C_2, C_3 be circles in general position. For each of the pairs C_1, C_2 and C_2, C_3 choose one of the two families of circles touching both circles; call these families $\mathcal{F}_1$ and $\mathcal{F}_2$. Consider the curve consisting of pairs of circles $(S_1, S_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ such that S_1 is tangent to S_2. According to [1], this curve has two irreducible components that are rational curves of bidegree (1, 1) and another irreducible com-

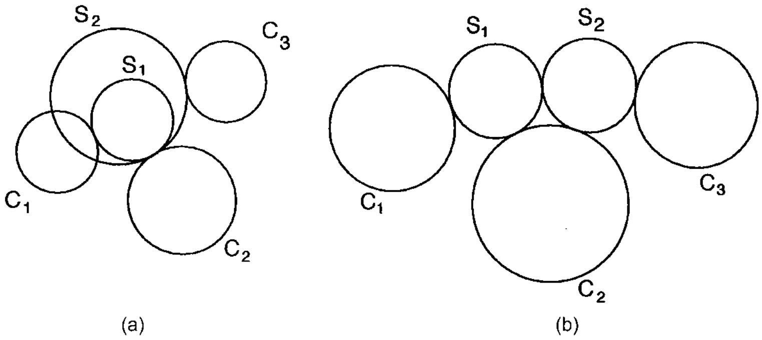

Figure 7.

ponent that is an elliptic curve of bidegree (2, 2) – see Figure 7(a) for the former and Figure 7(b) for the latter.

Let $C_1, \dots, C_n$ be circles in general position. For each $i = 1, \dots, n$ we choose one of the two families of circles, tangent to C_i and C_{i+1}; call the chosen family $\mathcal{F}_i$. One has elliptic curves $E_i \subset \mathcal{F}_i \times \mathcal{F}_{i+1}$; denote the projections on the factors by π_i and π_i'.

A collection of circles $C_1, \dots, C_n$, along with a choice of the families $\mathcal{F}_i$, will be called *nice* if for every i the following holds: the two Apollonius circles for C_{i-1}, C_i, C_{i+1} that lie in $\mathcal{F}_i$ but not in $\mathcal{F}_{i-1}$ coincide with the two Apollonius circles for C_i, C_{i+1}, C_{i+2} that lie in $\mathcal{F}_i$ but not in $\mathcal{F}_{i+1}$. Figure 8 shows a nice collection of 4 circles (the family $\mathcal{F}_i$ contains the circles that touch and contain both circles C_i and C_{i+1}). The niceness condition on n circles imposes $2n$ relations: each consecutive circle C_i is tangent to two circles determined by the preceding three circles $C_{i-1}, C_{i-2}, C_{i-3}$. It follows that the variety of nice n-tuples of circles is (at least) n-dimensional. Note also that niceness does not impose any restrictions on a triple of circles: given a choice of $\mathcal{F}_1$ and $\mathcal{F}_2$, one can choose $\mathcal{F}_3$ so that the triple is nice.

The next result is an extension of the Money-Coutts theorem as formulated in [1] and, at the same time, of Theorem 1(a).

THEOREM 2. *Let $\{C_i, \mathcal{F}_i\}$ be a nice collection of n circles. Suppose that circles $S_1 \in \mathcal{F}_1, \dots, S_n \in \mathcal{F}_n, S_{n+1} \in \mathcal{F}_1$ are chosen such that $(S_1, S_2) \in E_1, \dots, (S_n, S_{n+1}) \in E_n$. Then it is possible to choose circles $S_{n+2} \in \mathcal{F}_2, \dots, S_{2n} \in \mathcal{F}_n, S_{2n+1} \in \mathcal{F}_1$ such that S_{2n+1} coincides with S_1.*

Proof. We repeat the proof from [1]. Consider the two projections: π_{i-1}': $E_{i-1} \to \mathcal{F}_i$ and π_i: $E_i \to \mathcal{F}_i$. The branch points of the first projection are the two null-circles of the pair C_i, C_{i+1} and the two Apollonius circles for C_{i-1}, C_i, C_{i+1} that lie in $\mathcal{F}_i$ but not in $\mathcal{F}_{i-1}$; the branch points of the second projection are again the two null-circles of the pair C_i, C_{i+1} and the two Apollonius circles for C_i, C_{i+1}, C_{i+2} that lie in $\mathcal{F}_i$ but not in $\mathcal{F}_{i+1}$. Due to niceness, the branch points of π_{i-1}' and π_i coincide.

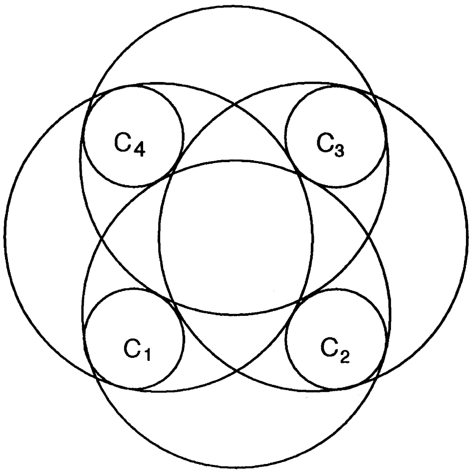

Figure 8.

Therefore there are isomorphisms $\phi_1 \colon E_1 \to E_2, \phi_2 \colon E_2 \to E_3, \ldots, \phi_n \colon E_n \to E_1$ such that $\pi_{i+1}\phi_i = \pi_i'$ for all i. Identify $E_1 = \cdots = E_n := E$ and $\pi_i' = \pi_{i+1}$ by means of $\phi_i,\ i = 1, \ldots, n-1$. Then ϕ_n becomes an automorphism τ of E that, with a suitable choice of $\phi_1, \ldots, \phi_{n-1}$, can be assumed a translation (and not an involution).

Let σ_i be the involution of E that interchanges the preimages of π_i. One has: $S_1 = \pi_1(e)$ for some $e \in E$. Then $S_2 = \pi_1'(\alpha_1 e)$ where α_1 is either the identity or σ_1. Continuing the same way, one has $S_{n+1} = \pi_n'(\alpha_n \cdots \alpha_1 e) = \pi_1(\tau\alpha_n \cdots \alpha_1 e)$ with $\alpha_j \in \{1, \sigma_j\}$.

Similarly, to determine circles $S_{n+2}, \ldots, S_{2n+1}$ with $(S_j, S_{j+1}) \in E$, one chooses $\beta_j \in \{1, \sigma_j\},\ j = 1, \ldots, n$ and sets $S_{n+1+j} = \pi_j'(\beta_j \cdots \beta_1 \tau \alpha_n \cdots \alpha_1 e)$, in particular, $S_{2n+1} = \pi_1(\tau\beta_n \cdots \beta_1 \tau \alpha_n \cdots \alpha_1 e)$. The choice is made as follows: $\beta_j = \alpha_j$ for $j = 2, \ldots, n$, and β_1 is chosen so that $\beta_n \cdots \beta_1$ is an involution. Then one has $\tau\beta_n \cdots \beta_1 \tau \alpha_n \cdots \alpha_1 = \beta_n \cdots \beta_1 \alpha_n \cdots \alpha_1 = (\alpha_n \cdots \alpha_2 \beta_1)^2 \beta_1 \alpha_1 = \beta_1 \alpha_1$. It follows that $S_{2n+1} = \pi_1(\beta_1 \alpha_1 e) = S_1$, and we are done.

Remarks. (1) The niceness condition on polygon P appeared in Theorem 1 as a technical condition needed in the proof, namely, to provide formula (2); the algebro-geometrical approach gives a more conceptual explanation of this condition.

(2) Consider the situation of Theorem 1(a): P is a nice n-gon with odd n, the choice of the families $\mathcal{F}_i$ being determined by the choice of the circles S_i in Theorem 1. Using the same notation as above, one has the isomorphisms $\psi_i \colon E_i \to E_{i+1}$ that take (S_i, S_{i+1}) to (S_{i+1}, S_{i+2}). The through automorphism $\psi_n \cdots \psi_1 \colon E_1 \to E_1$ is an involution of the elliptic curve, and its second iteration is the identity. In particular, it follows that a nice polygon in Theorem 1(a) can be per-

turbed by replacing its sides by arcs of (sufficiently large) circles that make a nice collection.

(3) Unlike the case of an odd n, Theorem 1(b) is not a particular case of Theorem 2: the choice of the circles S_i in the proof of the latter is not the same as that in the formulation of the former. It would be interesting to find a generalization of Theorem 1(b) along the lines of Theorem 2.

(4) It is possible, *a priori*, that the translation τ in the proof of Theorem 2 is the identity. Can such a poristic case occur for odd n? The answer is negative for $n = 3$ unless the configuration of circles is very degenerate, namely, they belong to a pencil – see [4].

Acknowledgement

I am grateful to W. Barth, M. Berger, J. Duncan and Ch. Goodman-Strauss for their interest and help. The work was supported in part by an NSF grant.

References

1. Barth, W. and Bauer, Th.: Poncelet theorems, *Exposition. Math.* **14** (1996), 125–144.
2. Belen'ky, V. and Zaslavsky, A.: On the Malfatti problem, *Kvant* **4** (1994), 38–42.
3. Evelyn, C., Money-Coutts, G. and Tyrrell, J.: *The Seven Circles Theorem and Other New Theorems*, Stacey Int., U.K., 1974.
4. Tyrrell, J. and Powell, M.: A theorem in circle geometry, *Bull. London Math. Soc.* **3** (1971), 70–74.