Arnold's Problems
by Vladimir I. Arnold (ed.)
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REVIEWED BY SERGEI TABACHNIKOV

This is a very unusual book, and I think it will be appropriate to put things into perspective and to provide some background information.

V. I. Arnold is a famous mathematician who belongs to the generation of Russian mathematicians, now approaching their 70-th anniversaries. I will not list here Arnold's numerous awards and prizes: such lists are easily available, in particular, on the Internet. I would like to mention a most unusual distinction: there is a small planet, Vladarnolda, discovered in 1981 and registered under # 10031, named after V. Arnold.

This generation was extremely rich with talent and includes such towering figures as D. V. Anosov, A. A. Kirillov, Yu. I. Manin, S. P. Novikov, Ya. G. Sinai and others. One cannot help noticing that these mathematicians were born in about 1937, the year of Great Terror in Soviet Union. I once asked V. Arnold why, in his opinion, this generation was so exceptional. His explanation was that the preceding generation of Soviet mathematicians largely perished in the Second World War. As a result, his generation had more freedom to develop independently, growing like trees on a cleared land, not in the shade of taller trees in a forest. One should also take into account that this generation came of age in the relatively liberal time of the post-Stalin "thaw".

Moscow was, and still remains, one of the leading mathematical centers of the world (I am talking only about Moscow, and only about the period from the late 1950s, because that is where and when most of the activity of V. Arnold and his seminar took place.) Mathematical life evolved around a number of seminars, some of them quite legendary. Mathematics of the highest quality and charismatic personalities of the organizers made these seminars strong attractors for budding mathematicians. (Probably, the most influential and famous of all was Gelfand's Seminar; see [12, 13, 14, 15, 16, 17, 18, 19, 20] for collection of papers representing the Arnold, Gelfand, Kirilllov, Manin, Novikov and Sinai seminars). It is, of course, beyond the scope of this review to describe Moscow mathematical life of the period in any meaningful detail; the interested reader is recommended to the book [21].

Arnold's Seminar started to work in the 1960s; this is a 2 hour weekly event. The format is different from the majority of the mathematical seminars in the West that last an hour. This makes a substantial difference in the style of presentation: a 2 hour talk is likely to include proofs. Traditionally, the first meeting of every semester was devoted to Arnold's talk on open problems. These problems, along with comments, comprise the book under review. The problems were addressed to all the participants of the seminar: undergraduate students, graduate students and established researchers.

Concerning participation of undergraduate students, an explanation is in order: they specialized in their major from day one, taking a large number of mandatory courses in mathematics. Starting with the 3-rd year, undergraduate students of mathematics at Moscow State University had to choose an advisor and by the end of the last, 5-th, year to produce a research thesis. It was not unusual for undergraduate students to have published results in refereed journals.

In about 1990, the Iron Curtain fell, and many Soviet mathematicians left, temporarily or permanently, for the West. In particular, V. Arnold accepted a position in Paris where he was spending the spring semester every year and where the Paris branch of Arnold's Seminar started to work in 1993. This significantly expanded the geography of the Seminar: many of its former participants, now working in various countries of the world, continue to follow Arnold's problem lists, available, in particular, on the Internet.

Before reviewing the content of the book, a few words about V. Arnold's views on how to practice and to teach mathematics are in order; this philosophy is implicit in the book and it is explicitly expressed, in particular, in the interviews given by Arnold to various mathematical magazines [3, 6, 8, 11]; see also [1, 2]. I will quote from these sources.

What is mathematics? "Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap" [1]. "It seems to me that modern science (that is, theoretical physics, along with mathematics) is a new kind of religion: the cult of truth founded by I. Newton 300 years ago" [8] (my translation into English).

On unity of mathematics: "Jacobi noted the most fascinating property of mathematics, that in it one and the same function controls both the presentation of an integer as a sum of four squares and the real movement of a pendulum" [1].

About fashion in mathematics: "Evolution of mathematics resembles fast revolution of a wheel, so that drops of water fly in all directions. Fashion is this stream that leaves the main trajectory in the tangential direction. These streams of imitation works are most noticeable, they constitute the main part of the total volume, but they die out soon after parting with the wheel. To keep staying on the wheel, one must apply effort in the direction perpendicular to the main flow" [8] (my translation into English).

It is worth mentioning that Arnold himself has created fashions in mathematics more than once!

On errors in mathematics: "Mistakes are an important and instructive part of mathematics, perhaps as important a part as the proofs. *Proofs are to mathematics what spelling (or even calligraphy) is to poetry.* Mathematical works do consist of proofs, just as poems do consist of words" [2].

On applications: "A remarkable property of mathematics, which one cannot help but admire, is the unreasonable effectiveness of the most abstract, and at first glance completely useless, of its branches, provided that they are beautiful" [11]. "According to Louis Pasteur, there exist no applied sciences – what do exist are *applications* of sciences" [6].

On advising students: "A student is not a sack to be filled but a torch to be lighted" [8] (my translation into English).

And finally, from author's preface to the Russian edition of Arnold's Problems (2000; reprinted in the book under review): "I. G. Petrovskii, who was one of my teachers in Mathematics, taught me that the most important thing that a student should learn from his supervisor is that some *question is still open.* Further choice of the problem from the set of unsolved ones is made by the student himself. To select a problem for him is the same as to choose a bride for one's son."

The book under review consists of two parts: the first third is occupied by formulations of the problems and the rest comprise comments to the problems. The problems are grouped by years, from 1956 to 2003. The total number of problems is 861; some of them repeat, usually, with variations (in the case of such "twin" problems, only one comment is given, with reference to other similar problems). In the late 1950s and in the 1960s, the problems were not collected systematically, and many were lost. From 1970 on, the lists are more-or-less complete. The number of problems per year changes significantly, with the annual average in 1970–2003 of about 24. Some of the problems were posed by other mathematicians, in which cases the name of the proposer is indicated.

The formulations of the problems in the book are usually not in the most general form, but rather consider the first non-trivial case of a general phenomenon; to discover and describe this general phenomenon is a substantial part of the problem (V. Arnold refers to this the "Russian style" of posing problems).

Working on this review, I tried to classify the problems according to the areas of mathematics they represent. This attempt of "subject classification" proved fruitless. It is fair to say that the majority of problems reflect long-term research interests of V. Arnold, and these interests span a sizable part of contemporary mathematics.

Many problems concern dynamical systems, in particular, the KAM (Kolmogorov-Arnold-Moser) theory, local and global singularity theory, real algebraic geometry, symplectic and contact geometry, classical mechanics, topological hydrodynamics. Numerous problems in the late 1980s and 1990s concern geometrical and topological, in particular, symplectic and contact, generalizations of the classical four vertex theorem (*a plane oval has at least four curvature extrema*; this result, due to S. Mukhopadhyaya, 1909, has generated a huge literature), and a substantial part of the problems in the recent years belongs to number theory, in particular, to the theory of multi-dimensional continued fractions.

A typical formulation of a problem is one paragraph long – in some cases, just a sentence – but there are notable exceptions to this rule, and all the problems in 2003 have much more detailed formulations, sometimes several pages long (but they are not commented upon). Instead of continuing with this general description, necessarily boring, let me give a few examples.

The first three are examples of prophetic vision: these questions and their generalizations and ramifications significantly influenced the development off contemporary mathematics.

Problem 1958-1. Let us consider a partition of the closed interval $[0;1]$

into three intervals $\Delta_1, \Delta_2, \Delta_3$ and rearrange them in the order $\Delta_3, \Delta_2, \Delta_1$. Explore the resulting dynamical system $[0;1] \to [0;1]$: is it true that the mixing rate and similar ergodic characteristics are the same for almost all lengths $(\Delta_1, \Delta_2, \Delta_3)$ of the partition interval? An analogous question may be asked for n intervals and for arbitrary permutations as well (changing the orientation of some intervals also being allowed).

The theory of interval exchange transformations and the related subjects of flat surfaces, quadratic differentials and billiards in rational polygons have become fast growing and important areas of research, rich with deep and beautiful results.

Problem 1963-1. Is there true instability in multidimensional problems of perturbation theory where the invariant tori do not divide the phase space?

This problem concerns the phenomenon known as "Arnold diffusion" in the theory of small perturbations of integrable Hamiltonian systems (KAM theory). A breakthrough was recently made by J. Mather who proved the existence of Arnold diffusion in some cases.

Problem 1966-4. Let a diffeomorphism $A : q \mapsto q + f(q)$ of the torus $T^2 = \{(q_1, q_2) \ mod \ 2\pi\}$ preserve the measure $dq_1 \wedge dq_2$ and the center of mass:

$$\int_{T^2} f(q) dq_1 dq_2 = 0.$$

Prove that A has at least 4 fixed points counting multiplicities and at least 3 geometrically distinct fixed points.

This is one of the celebrated Arnold conjectures on fixed points of exact symplectomorphisms; it was proved, in 1983, by Conley and Zehnder. By now, symplectic topology has grown into a large and rapidly developing field, one of the main achievements in mathematics of the last 20 years.

Let me give a few more samples of problems, from very broad to fairly concrete.

Problem 1979-4. Construct a "complexification" of the homology theory (replacing a boundary with a two-sheet branched covering). What is the complexification of orientation? (Apparently, it assigns an element of $\mathbf{Z} = \pi_1(U(n))$ to a loop?)

Problem 1987-14. Do there exist smooth hypersurfaces in $\mathbf{R}^n$ (other than the quadrics in odd-dimensional spaces), for which the volume of the segment cut by any hyperplane from the body bounded by them is an algebraic function of the hyperplane? *For these quadrics the volume is an algebraic function (Archimedes), and the area of segments of plane curves is never algebraic (Newton).*

Problem 1989-18. The sequence of meandric numbers $1, 1, 2, 3, 8, 14, 42, 81, \dots$ is defined as follows. Suppose an infinite river running from south-west to north-east intersects an infinite straight road going from the west to the east under n bridges numbered $1, \dots, n$ in the order from west to east. The order of the bridges along the river determines a *meandric permutation* of the numbers $1, \dots, n$. The *meandric number* M_n is the number of meandric permutations on n elements. *Meandric numbers possess many remarkable properties; for example, M_n is odd iff n is a power of 2 (S. K. Lando).* Find the asymptotics of M_n as $n \to \infty$. *It is known that $c4^n < M_n < C16^n$ for some constants c, C.*

Problem 1994-17. Find all projective curves equivalent to their duals. *The answer seems to be unknown even in* $\mathbf{RP}^2$.

I would like to add, in case the reader has doubts, that there exist even smooth convex curves, other than conics, projectively equivalent to their duals!

Problem 2000-7. There are observations that the number of the species (of animals, insects, birds,...) on an island of area S is proportional to $S^{1/4}$, whereas the number of the cells types in an organism with the genome of N genes grows with N like $N^{1/2}$. How can one explain these exponents? Compare with the Kolmogorov law, according to which the radius of the minimal but still typical brain or computer of N elements grows like $N^{1/2}$ (rather than like $N^{1/3}$, as the volume argument suggests).

I cannot help mentioning the very first problem in the book, # 1956-1, "The rumpled dollar (originally, rouble) problem": is it possible to increase the perimeter of a rectangle by a sequence of foldings and unfoldings?

As many problems in the book, this one has an interesting story to it. The problem has become part of mathematical folklore, it is also known as the "Margulis napkin problem". The question is answered in the affirmative, see [10, 9]. It is interesting that the problem was solved by origami practitioners way before it was posed (at least, in 1797, in the Japanese origami book "Senbazuru Orikata"), see [23] and [4, 5]. One cannot help but agree with M. Berry's law: *Nothing is ever discovered for the first time* (posted on his web site)!

The comments to the problems vary from detailed surveys several pages long, equipped with extensive bibliography, to very brief, one paragraph, references to the literature. Sometimes a comment simply states: "Nothing is known". Each comment is marked $\mathcal{H}$ or $\mathcal{R}$, indicating a historic or research comment. Some problems are commented upon more than once, by a number of authors.

The list of authors of the comments consists of 59 names; the number doubled compared with the 2000 Russian edition. Author index for comments is provided; all but one (J. Lagarias) are current or former participants of Arnold's Seminar, his students, in the general sense of the word (this includes the author of this review); one of the most prolific commentators is V. Arnold himself. Many of the commentators contributed to (partial) solutions of the problems presented in the book (according to Arnold, the average half-life of a problem is 7 years).

The genre of the book is rather rare; "The Scottish Book" [22] comes to mind (a collection of about 200 problems, with comments, composed by a group of Lvov mathematicians in 1935–41; this group included Banach, Mazur, Steinhaus, Ulam and others.). Surely, many mathematical problem lists are known, some of them very influential (for example, Hilbert's problems), but none come close in their sheer volume, width and breadth to the one under review. Arnold's problems remain today as inspiring and stimulating as ever, and the book belongs to every mathematical library and the bookshelf of every research mathematician.

The authors, editors and publishers of the book did a fantastic and very difficult job. I do hope that this is an ongoing project, and there will be further editions, with new problems and new comments. I would like to suggest that the new edition(s) have a broader commentator base; the creators of the book may consider having a designated web site for downloading comments on the problems. It would be very helpful to have some kind of a problem index and their rough classification by topics, but as I already mentioned, this is a very non-trivial task.

In conclusion I would like to refer to another review of the same book [7], written by an editor of the Russian edition of "Arnold's Problems" and a major contributor to the comments, M. Sevryuk.

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