Differential topology

Lecture notes

1. Smooth manifolds and smooth maps

1.1. Let $U \subset \mathbf{R}^k, V \subset \mathbf{R}^n$ be open sets. A map $f : U \to V$ is called *smooth* if its every component (and there are n) is an infinitely differentiable function.

1.2. Let $X \subset \mathbf{R}^k, Y \subset \mathbf{R}^n$ be arbitrary sets. A map $f : X \to Y$ is called smooth if each point $x \in X$ has an open neighborhood U and there is a smooth map $F : U \to \mathbf{R}^n$ with $F|_X = f$.

The obvious properties hold (formulate and prove some): e.g., the composition of smooth maps is smooth.

1.3. A *diffeomorphism* $f : X \to Y$ is a homeomorphism such that both f and f^{-1} are smooth.

Note a (potential counter)example: $X = Y = \mathbf{R}, f(x) = x^3$.

1.4. $M \subset \mathbf{R}^k$ is a *smooth manifold* if each point $x \in M$ has an open neighborhood $U \subset M$ which is diffeomorphic to an open set $V \subset \mathbf{R}^m$. The diffeomorphism $g : U \to V$ is called a coordinate system and its inverse a parameterization. The number m is the dimension of M^m (thus the notation).

This formal definition means what one already knows: a smooth manifold is something without corners or boundary. From now on when we say "map" we mean "smooth map"; when we say "manifold" we mean "smooth manifold".

Examples. The unit sphere in $\mathbf{R}^{k+1}$ is a k-dimensional manifold. To prove that consider the north pole and show that the projection of its neighborhood on the equatorial hyperplane is a coordinate system. Another example: if M^m and N^n are smooth manifolds then so is $M \times N$, and its dimension is $m + n$ (prove it!)

Digression. One may define a smooth manifold without an ambient Euclidean space as a reasonable topological space with an open covering U_i, homeomorphisms $g_i : U_i \to V_i \subset \mathbf{R}^m$ such that when the sets U_i and U_j intersect the map $g_i g_j^{-1}$ is smooth (the latter maps are called transition functions). This definition is more general but essentially equivalent to the one above. One may define diffeomorphism similarly (do it yourself).

Exercises. (i) Cover the circle by two open sets, the complements to the poles. Consider the two stereographic projections, f and g, from the poles to $\mathbf{R}$. Compute the transition function fg^{-1}.

(ii) The antipodal map of the unit sphere is a diffeomorphism.

(iii) The open unit ball is diffeomorphic to the whole space.

2. Tangent spaces and differentials

2.1. For open $U \subset \mathbf{R}^k$ the tangent space $T_x U$ at each point $x \in U$ is defined to be $\mathbf{R}^k$.

2.2. For $U \subset \mathbf{R}^k$ and $f : U \to \mathbf{R}^n$ define the *differential* (or derivative, which is the same) $d_x f : \mathbf{R}^k \to \mathbf{R}^n$ by the (familiar) formula:

$$d_x f(u) = \lim_{t \to 0} \frac{f(x + tu) - f(x)}{t};$$

here $x \in U$, and $d_x f$ is a linear map for each x.

Again the obvious properties hold: chain rule, the differential of a linear map is the map itself (formulate and prove!).

2.3. Dimension is invariant under diffeomorphisms.

Claim. *If an open $U \subset \mathbf{R}^k$ is diffeomorphic to an open $V \subset \mathbf{R}^n$ then $k = n$.*

Proof. On the level of the differentials one has a linear isomorphism between $\mathbf{R}^k$ and $\mathbf{R}^n$ (fill out the missing details!).

2.4. Let $M^m \subset \mathbf{R}^k$ be a manifold, and x is a point of M. We want to define the *tangent space* $T_x M$. Intuitively, this is the linear m-dimensional space that approximates M at x. Choose a parameterization $g : U \to M$ where $U \subset \mathbf{R}^m$; let $g(u) = x$. Then define $T_x M$ as the image of the differential $d_u g$.

Claim. *(i) The definition is correct, i.e., does not depend on the choice of parameterization. (ii) dim $T_x M = m$.*

Proof is an exercise (done in class).

2.5. Consider two manifolds $M \subset \mathbf{R}^k$ and $N \subset \mathbf{R}^l$ and a map $f : (M, x) \to (N, y)$. We define the differential (linear map for each x)

$$d_x f : T_x M \to T_y N$$

as follows. By definition of a smooth map there is an open set $W \subset \mathbf{R}^k$ and a smooth map $F : W \to \mathbf{R}^l$ such that $F|_M = f$. Set $d_x f = d_x F|_{T_x M}$.

Claim. *(i) The definition is correct, i.e., does not depend on the choice of F. (ii) The range of $d_x f$ is indeed $T_y N$.*

Proof again is an exercise (also done in class).

Corollary. *If M^m is diffeomorphic to N^n then $m = n$.*

The proof repeats the one in sect. 2.3.

3. Regular and singular values

3.1. Given a map $f : M^n \to N^n$, a point $x \in M$ is called *regular* if the linear map $d_x f$ is non-degenerate. By the implicit function theorem f is a diffeomorphism in a neighborhood of x. A point is *singular* if it is not regular. A point $y \in N$ is called a *regular value* if its preimage $f^{-1}(y)$ consists only of regular points or is empty, and a *singular value* otherwise.

Claim. *If M is compact and $y \in N$ is a regular value then $f^{-1}(y)$ is a finite set.*

Proof. The set $f^{-1}(y)$ is closed since f is continuous. A closed subset of a compact is compact. For each $x \in f^{-1}(y)$ the map f is a diffeomorphism of a sufficiently small neighborhood of x to a neighborhood of y. Therefore the set $f^{-1}(y)$ is discrete. A discrete and compact set is necessarily finite.

Denote the cardinality of the set $f^{-1}(y)$ by $\#f^{-1}(y)$. This is a function of y, locally constant on the set of regular values.

3.2. **Fundamental theorem of algebra**. This is the famous statement that every complex polynomial $f(z) = z^n + a_1 z^{n-1} + ... + a_n$ has a root.

Consider f as a map from $\mathbf{C}$ to itself. Then (i) f is smooth (prove!).

Extend f to the Riemann sphere $\mathbf{S}^2$ by $f(\infty) = \infty$. Then (ii) the extended f is still smooth (prove).

Moreover, (iii) the singular points of f are the roots of the derivative f' (prove).

There are finitely many such roots (at most $n-1$), thus f has finitely many singular values. The complement to this finite set consists of regular values, and the function $\#f^{-1}(y)$ is constant on this open set. This constant value is not zero: the polynomial f has some values; thus every point of the sphere is a value, in particular, this hold for the value zero.

3.3. Let us generalize the notion of regular/singular point. Given a map $f : M^m \to N^n$, a point $x \in M$ is called singular if rk $d_x f < n$. For example, if $m < n$ then all points are singular.

The set of singular points can be big in M but its image in N is always small.

Theorem (Sard). *Let $U \subset \mathbf{R}^m$ be an open set, $f : U \to \mathbf{R}^n$ be a smooth map, and $S \subset U$ the set of singular points. Then $f(S)$ has zero measure.*

It follows that $\mathbf{R}^n - f(S)$ is dense, so almost every $y \in \mathbf{R}^n$ is a regular value.

A proof of the Sard theorem is rather technical: see your notes or look it up in a book.

3.4. In this section we give a method of constructing smooth manifolds.

Proposition. *Given a smooth map $f : M^m \to N^n$, let $y \in N$ be a regular value. Then $f^{-1}(y)$ is a smooth $(m-n)$-dimensional manifold.*

Of course this is meaningful only for $m \geq n$.

Proof. We have: $M \subset \mathbf{R}^k$. Consider $x \in f^{-1}(y)$ and let $K = \text{Ker } d_x f$. Surely K will be the tangent space to $f^{-1}(y)$ at x. Let π be a projection $\mathbf{R}^k \to K$, and consider the map $F : M \to N \times K$ given by the formula $F(x) = (f(x), \pi(x))$. Then F is a map of manifolds of the same dimensions, and $d_x F = (d_x f, \pi)$. Thus $d_x F$ is non-degenerate, and F is locally a diffeomorphism. Note that $F(f^{-1}(y)) = y \times K$ is a smooth manifold, therefore so is $f^{-1}(y)$.

A typical example: the unit sphere is a smooth manifold since it is the preimage of a regular value 1 of the function $x_1^2 + ... + x_n^2$.

4. Manifolds with boundary

4.1. A smooth manifold is modeled on an open set in Euclidean space; a manifold with boundary on an open subset of a closed half-space.

Let $\mathbf{R}^m_+$ be the upper half-space $x_m \geq 0$; its boundary $\partial \mathbf{R}^m_+$ is defined as $\mathbf{R}^{m-1} = \{x_m = 0\}$. Then $M^m \subset \mathbf{R}^k$ is called a *manifold with boundary* if each point $x \in M$ has a neighborhood that is diffeomorphic to an open subset in $\mathbf{R}^m_+$. The *boundary* ∂M is the set of points that go to $\partial \mathbf{R}^m_+$ under such diffeomorphisms.

Claim. *∂M is a smooth $(m-1)$-dimensional manifold (without boundary).*

Prove it yourselves.

Thus $\partial^2 = 0$ which should ring the bell for those who took algebraic topology.

Exercise. Let $f : M \to \mathbf{R}$ be a smooth function and 0 be a regular value. Then $f^{-1}[0, \infty] \subset M$ is a smooth manifold with boundary $f^{-1}(0)$.

For example, the unit ball is a manifold whose boundary is the unit sphere.

An analog of the proposition in the previous section holds as well.

Proposition. *Let M be a manifold with boundary and $f : M^m \to N^n$ is a smooth map. Let $y \in N$ be a regular value for f and for $f|_{\partial M}$. Then $f^{-1}(y)$ is a manifold with boundary and $\partial f^{-1}(y) = f^{-1}(y) \cap \partial M$.*

4.2. **The Brouwer fixed point theorem** is one of the most celebrated results in topology; it says that a continuous map of a closed ball has a fixed point. Usually it is proved using the machinery of algebraic topology. Here we will prove a smooth analog of this result.

Lemma. *Let M be a compact manifold with boundary. Then there does not exist a smooth map $f : M \to \partial M$ that leaves every point of the boundary fixed.*

Proof. Assume such a map exists and let $y \in \partial M$ be its regular value. Since y is also a regular value of the restriction of f to the boundary (which is the identity) the set $f^{-1}(y)$ is a smooth compact 1-dimensional manifold whose boundary is y (see proposition in the previous section). But there are only two compact 1-dimensional connected manifolds: a circle (no boundary) and a segment (the boundary consists of two points). This is a contradiction.

An important particular case: the identical map of the sphere cannot be extended to the ball.

Now we prove Brouwer's theorem. Let f be a map of the unit ball $\mathbf{D}^n$ without fixed points. Define a map $g : \mathbf{D}^n \to \mathbf{S}^{n-1}$ as follows: $g(x)$ is the point of the sphere on the line $(x, f(x))$ which is closer to x than to $f(x)$. Then g is a smooth map (prove it!) which leaves every point of the sphere fixed. This violates the above lemma.

One can deduce the continuous fixed point theorem from its smooth version since every continuous map can be approximated by smooth ones (actually, even by polynomials).

4.3. **Application: a Frobenius theorem**. This theorem says: a matrix with all non-negative entries has a non-negative eigenvalue.

Let A be such a matrix considered as a linear transformation of $\mathbf{R}^n$. Then A sends the (closed) positive ortant to itself. WLOG, A is non-degenerate. Then the map $f : v \to Av/|Av|$ is a smooth map of the unit sphere to itself sending the closed positive ortant of the sphere to itself. This positive ortant is topologically the $n-1$-dimensional ball. By Brouwer's theorem f has a fixed point, and this is an eigenvector with a positive eigenvalue.

5. Immersions and embeddings

5.1. The following terminology is standard. An *immersion* is a map $f : M^m \to N^n$ such that $d_x f$ is injective for all $x \in M$. An *embedding* is an immersion that is also a homeomorphism on its image; for compact manifolds this simply means one-to-one. A *submersion* is a map such that $d_x f$ is surjective for all $x \in M$.

Examples. An 8-shaped curve is an immersion of a circle to the plane; an embedding of a circle in the plane is a simple closed curve. The projection of $M \times N$ onto M is a surjection. Every point of a surjection is regular.

5.2. By definition a manifold M^m is a subset in some Euclidean space. The inclusion $M \subset \mathbf{R}^k$ is an embedding (prove it!). The dimension k can be very big a priori; how small can one make it?

Theorem (Whitney). *Every m-dimensional manifolds can be embedded into $\mathbf{R}^{2m}$.*

We will prove an easier result with $2m$ replaced by $2m+1$ and for compact manifolds.

Proof. Let $k > 2m+1$; we will show that there is a linear projection $\pi : \mathbf{R}^k \to \mathbf{R}^{k-1}$ such that $\pi(M)$ is embedded. This will prove the result inductively.

When does $\pi(M)$ fail to be embedded? Either if it is not immersed or when $\pi|_M$ is not one-to-one. The former happens if Ker π is tangent to M at some point. Consider the map $f : TM \to \mathbf{R}^k$ given by $f(x, v) = v$ (more accurately, $f(x, v) = d_x i(v)$ where i is the inclusion $M \subset \mathbf{R}^k$). Since f is smooth (why?) and dim $TM = 2m$, by Sard's theorem almost every vector of $\mathbf{R}^k$ is not in the image of f. Let w be such a nonzero vector; then the projection along w sends M to an immersed manifold.

Likewise, consider the map $g : M \times M \times \mathbf{R} \to \mathbf{R}^k$ given by $g(x, y, t) = t(x - y)$. Again almost every vector of $\mathbf{R}^k$ is not in the image, and if w is such a nonzero vector then the projection along w is one-to-one on M. Choosing w satisfying both conditions provides the desired projection.

Note that the first part of the proof provides an immersion $M^m \to \mathbf{R}^{2m}$. Also note that the map f is the proof is the limit of the map g as $x \to y$.

6. Degree mod 2

Consider a map $f : M^n \to N^n$ and let $y \in N$ be a regular value. How does $\#f^{-1}(y)$ depend on the point y? In this section we will see that it is independent of y mod 2.

6.1. Two maps $f, g : M \to N$ are called *smoothly homotopic* if there exists a smooth map $F : M \times [0, 1] \to N$ such that $F(x, 0) = f(x)$ and $G(x, 1) = g(x)$ for all $x \in M$. The map is called a *smooth homotopy* between f and g. Notation: $f \sim g$.

Lemma. *Smooth homotopy is an equivalence relation.*

Prove it yourselves.

Similarly, let f and g be diffeomorphisms of a manifold M. They are called *smoothly isotopic* if there exists a smooth homotopy $F : M \times [0,1] \to M$ such that for every t the map $F(\ ,t)$ is a diffeomorphism.

6.2. Let $f, g : M^n \to N^n$ be two maps, M is compact and without boundary (*closed*, for short). Let $y \in N$ be a regular value for both maps.

Lemma. *If $f \sim g$ then $\#f^{-1}(y) = \#g^{-1}(y)$ mod 2.*

Proof. Consider a homotopy $F : M \times [0,1] \to N$. Assume first that y is a regular value for F. Then $F^{-1}(y)$ is a 1-dimensional compact manifold with boundary $(f^{-1}(y) \times 0) \cup (g^{-1}(y) \times 1)$. Thus the cardinality of the boundary is $\#f^{-1}(y) + \#g^{-1}(y)$. The number of boundary points of a 1-dimensional manifold is always even, and we are done in this case.

Assume now that y is a singular value for F. Being a regular value of f and g the point y has a neighborhood U in which $\#f^{-1}(z)$ and $\#g^{-1}(z)$ are constant. Almost every point $z \in U$ is a regular value of F, and we can apply the preceding argument to such a regular value.

6.3. Every two points of a connected smooth manifold are "born equal". More precisely, the next result holds.

Lemma. *Let M be a connected manifold and $x, y \in M$. Then there exists a diffeomorphism f that takes x to y. Moreover, f can be chosen isotopic to the identity.*

Outline of Proof. It suffices to learn how to move a point by a diffeomorphism slightly. This can be done in a neighborhood diffeomorphic to an open ball in Euclidean space, and the point may be chosen the center of the ball. It is easy to move the center of a ball radially by a diffeomorphism, isotopic to the identity.

6.4. Put all this together.

Theorem. *Let M be a closed manifold, $f : M^n \to N^n$ a smooth map. If x and y are regular values of f then $\#f^{-1}(x) = \#f^{-1}(y)$ mod 2. Moreover, this residue mod 2 does not change if f is replaced by a homotopic map.*

The above residue mod 2 is called *the degree* of f mod 2 and denoted by $\deg_2(f)$.

Proof. Consider a diffeomorphism h of N, isotopic to the identity, that takes x to y. Then y is a regular value for $g = hf$. Note that $f \sim g$. By Lemma 6.2 $\#f^{-1}(y) = \#g^{-1}(y)$ mod 2. Notice also that $\#g^{-1}(y) = \#f^{-1}(x)$. Combining the two equalities we obtain the first claim. Let $f \sim g$. Then there is a common regular value $y \in N$, and the second claim follows from Lemma 6.2.

Exercises. (i) Consider the map of the unit circle given, in the complex notation, by $z \to z^n$. What is its degree mod 2?

(ii) Consider a complex polynomial of degree n as a smooth map of the Riemann sphere. Find its degree mod 2.

6.5. **Application**. Compare two maps of a closed manifold M to itself: the identity and a constant map. The degree for the former is 1 and for the latter it is 0. Therefore these maps are never homotopic.

Exercise. Is the same true for a manifold with boundary? An open manifold?

Corollary. *There does not exist a smooth map of a ball* $\mathbf{D}^n$ *to its boundary* $\mathbf{S}^{n-1}$ *whose restriction to the sphere is the identity.*

Proof. If such a map f existed then one would have a homotopy of the identity and a constant map: $F(x,t) = f(tx)$.

7. Orientation of manifolds

7.1. We assume the concept of orientation of a vector space is known; in particular, $\mathbf{R}^n$ has a standard orientation.

An *orientation* of a smooth manifold M^n is a choice of orientation of every tangent space T_xM satisfying the next condition: for each $x \in M$ there is a neighborhood $U \subset M$ and an *orientation-preserving* diffeomorphism (coordinate system) h of U to an open set in $\mathbf{R}^n$; this means that d_xh is an orientation-preserving linear map for all x.

M is called *orientable* if there exists an orientation; if M is orientable and connected there are two opposite orientations.

7.2. If M has a boundary then there are 3 kinds of tangent vectors at a boundary point: inward, tangent to the boundary and outward. An orientation of a manifold determines an orientation of the boundary: given $x \in \partial M$, choose a positively-oriented basis $(e_1, ..., e_n)$ in T_xM such that $e_2, ..., e_n$ are tangent to the boundary and e_1 is an outward vector. Then the basis $(e_2, ..., e_n)$ gives the desired orientation of the boundary.

Examples. A sphere is the boundary of a ball; the ball being oriented, so is the sphere. The product of oriented manifolds is oriented (prove).

Exercise. The previous definition of the orientation of the boundary works for $n \geq 2$. What is the definition for $n = 1$?

7.3. **Remark**. One may define an orientation in terms of coordinate systems as follows. Let M be covered by open sets U_i and h_i are their maps to open sets in $\mathbf{R}^n$. One requests that if U_i and U_j intersect then the map $h_j^{-1}h_i$ is an *orientation-preserving* diffeomorphism of a domain in $\mathbf{R}^n$; this simply means that its Jacobian is positive.

Example. The Moebius band is not oriented (cover by two open neighborhoods and write down transition functions).

8. Degree of a map of oriented manifolds

If the manifolds involved are orientable then the degree of a map can be defined as an integer.

8.1. Let $f : M^n \to N^n$ be a map of oriented manifolds and $x \in M$ is a regular point. Define sign d_xf to be ± 1 depending on whether $d_xf : T_xM \to T_{f(x)}N$ preserves orientation or not. Let y be a regular value; define

$$\deg\ (f, y) = \sum_{x \in f^{-1}(y)} \text{sign } d_xf.$$

The main properties of degree are the same as those of degree mod 2.

Theorem. *The integer* $\deg(f, y)$ *does not depend on the choice of a regular value* y*. If* $f \sim g$ *then* $\deg f = \deg g$.

The proof follows the arguments in Section 6 adjusted to account for the orientation.

8.2. More precisely, one needs an analog of Lemma 6.2. First we prove

Lemma. *Let* X *be a compact manifold and* $M = \partial X$*, oriented as a boundary. If a map* $f : M \to N$ *extends to a map* $F : X \to N$ *then* $\deg\,(f, y) = 0$ *for every regular value* y*.*

Proof. As before we may assume that y is a regular value for F. Then $f^{-1}(y)$ is a compact 1-dimensional manifold in X. Let I be its component, diffeomorphic to an interval, and a and b its boundary points, $a, b \in M$. Claim:

$$\text{sign } d_a f + \text{sign } d_b f = 0.$$

Once this is done, summation over all components of $f^{-1}(y)$ yields the result.

To prove the claim, note that I can be given an orientation as follows. Let $x \in I$ and $e_1, ..., e_{n+1}$ be a positive basis of $T_x X$ such that e_1 is tangent to I. Then e_1 gives I the positive orientation if $d_x F$ sends $e_2, ..., e_{n+1}$ to a positive basis of TN.

Extend e_1 to each point of I; in particular, at one boundary point, say a, it has the outward direction, and at the other, i.e., b, the inward one. Then

$$\text{sign } d_a f = 1, \quad \text{sign } d_b f = -1,$$

and we are done.

An analog of Lemma 6.2 reads as follows.

Lemma. *If* $f \sim g$ *then* $\deg\,(f, y) = \deg\,(g, y)$.

Proof. One has a homotopy $F : M \times [0, 1] \to N$. The boundary of $M \times [0, 1]$ has two components: $M \times 0$ with the right orientation and $M \times 1$ with the wrong one. According to the previous lemma, $\deg F|\partial(M \times [0, 1]) = 0$. This degree equals $\deg\,(g, y) - \deg\,(f, y)$, and the result follows.

This completes the construction of the degree of a smooth map and the proves Theorem 8.1.

Examples. A constant map has degree 0. The identity has degree 1. An orientation-reversing diffeomorphism has degree -1. The map $z \to z^n$ of the unit circle has degree n.

9. Applications of degree

9.1. **Tangent vector fields on a sphere**.

A smooth *tangent vector field* on a manifold $M \subset \mathbf{R}^k$ is a smooth map $v : M \to \mathbf{R}^k$ such that for every $x \in M$ one has: $v(x) \in T_x M$.

Proposition. *An n-dimensional sphere has a non-vanishing tangent vector field if and only if n is odd.*

Proof. Such a field v provides a homotopy between the identity and the antipodal map: each point x moves along the great circle tangent to the vector $v(x)$.

Compute the degree of the antipodal map σ of the sphere. It is the composition of $n+1$ reflections in hyperplanes, each of which has degree -1 (why?) Thus deg $\sigma = (-1)^{n+1}$. It follows that if the field exists then n is odd.

Assume n is odd. Then $\mathbf{S}^n$ can be thought of as the unit sphere in $\mathbf{C}^k$, $n = 2k-1$. Let $v(x) = \sqrt{-1}x$. Then v is a smooth tangent vector field along $\mathbf{S}^n$ (prove it!).

Remark. It follows that for even n the *tangent bundle* $T\mathbf{S}^n$ is not *trivial*, that is, is not diffeomorphic to $\mathbf{S}^n \times \mathbf{R}^n$. In fact, $T\mathbf{S}^n = \mathbf{S}^n \times \mathbf{R}^n$ if and only if $n = 1, 3, 7$. The "if" part is easy and the "only if" part is hard.

One generalization is the following problem: how many linearly independent tangent vector fields are there on an n-dimensional sphere? This problem is completely solved (but the solution is quite hard). Another natural question is to describe those closed manifolds that admit non-vanishing tangent vector fields; this will be done shortly.

9.2. **Homotopy classification of maps to the sphere**.

If two maps $f, g : \mathbf{S}^1 \to \mathbf{S}^1$ are homotopic then deg f = deg g. The converse is true as well.

Proposition. *If deg f = deg g then $f \sim g$.*

Proof. A map $f : \mathbf{S}^1 \to \mathbf{S}^1$ can be lifted to a smooth function F such that $F(x+1) = F(x) + k$; a lifting is defined up to an integer. We may assume that $f(0) = 0$. Let F be a lifting such that $F(0) = 0$; let $k = F(1)$. Consider the function $H_k(x) = kx$; this function descends to a map $h_k : \mathbf{S}^1 \to \mathbf{S}^1$ and deg $h_k = k$. The linear homotopy $tF + (1-t)H_k$ implies that $f \sim h_k$. Therefore deg $f = k$, and every map f of degree k is homotopic to h_k. The result follows.

In fact, a more general result holds.

Theorem (Hopf). *Let M^n be a connected oriented closed manifold. Two maps $f, g : M \to \mathbf{S}^n$ are homotopic if and only if deg f = deg g.*

Outline of Proof. Assume first that for some regular value y the number of points in $f^{-1}(y)$ is equal to deg $f = k$. Let $x_1, ..., x_k$ be the preimages of y. Then one may deform f so that it maps neighborhoods U_i of x_i diffeomorphically onto $\mathbf{S}^n - N$, where N is the north pole, and maps the complement of these neighborhoods to N.

In general, $f^{-1}(y) = \{x_1, ..., x_{k+2m}\}$ where the signs at points $x_1, ..., x_k$ are the same and the signs at x_{k+i} and x_{k+m+i}, $i = 1, ..., m$ are pairwise opposite. As before, f is homotopic to a map that wraps neighborhoods of these points onto the sphere and sends the rest to N. It remains to "cancel" x_{k+i} with x_{k+m+i} by a homotopy of f: this is easily done since U_{k+i} and U_{k+m+i} wrap around the sphere in the opposite sense. As a result of these cancellations each map f of degree k is reduced to a canonical form, and the result follows.

9.3. **Rotation and winding numbers of plane curves**.

Given a closed immersed oriented plane curve $\gamma : \mathbf{S}^1 \to \mathbf{R}^2$, one can assign an integer to each component of its complement as follows. Fix a point x not on γ, and let y traverse γ. Then the vector xy can be normalized to a unit one, and one obtains a map from $\mathbf{S}^1$ to $\mathbf{S}^1$. The degree of this map is called the *rotation number* of the curve with respect to x and denoted by $r_\gamma(x)$. Clearly, $r_\gamma(x)$ does not change as long as x varies inside a component of γ. It is also clear that $r_\gamma(x) = 0$ if x lies in the unbounded component of the complement of the curve.

Lemma. *When x moves across γ the rotation number increases or decreases by 1, depending on the orientation of γ (recall the figure on the blackboard).*

Proof. Let x move across the curve to x', and let e be the unit vector proportional to xx'. Compute the rotation number at x and x' as the algebraic number of the preimages of e. The number of preimages at x is 1 greater, and the result follows.

Next, let y again traverse γ with unit speed. Then the velocity vector γ' determines a map from $\mathbf{S}^1$ to $\mathbf{S}^1$, called the (tangent) Gauss map. The degree of the Gauss map is called the *winding number* of the curve and denoted by $w(\gamma)$. The winding number does not change under a homotopy of an immersed curve (*regular homotopy*). Thus there are countably many regular homotopy classes of closed immersed plane curves.

Example. A counter-clockwise oriented circle with n identical kinks has $w = 1 \pm n$, depending on whether the kinks are clock or counter-clockwise.

Remark. A generic immersed plane curve does has neither triple points nor selftangencies. Two generic immersed plane curves are regularly isotopic if and only if they can be related by a sequence of triple point or selftangency moves.

A convenient method of computing the rotation and the winding numbers is as follows.

Lemma. *Resolve each double point according to the orientation of the curve (recall the figure); the curve γ becomes a collection of oriented circles. Then $w(\gamma)$ equals the number of positively oriented circles minus the number of negatively oriented ones, and $r_\gamma(x)$ is a similar algebraic number of the circles that contain x inside.*

Proof is an exercise (done in class).

The next question one asks is whether the winding number is the only obstruction to a regular homotopy between immersed plane curves. The answer is as follows.

Theorem (Whitney). *If two closed immersed plane curves have equal winding numbers then they are regularly homotopic.*

Proof. We will prove a slightly different version of the theorem concerning *long curves*, i.e., immersions of R^1 to R^2 which coincide with the horizontal axis off a sufficiently great disc.

First, one may introduce, by a regular homotopy, pairs of opposite kinks (recall the figure on the blackboard) so that γ is represented as a connected sum of a curve γ_0 with zero winding number and a standard curve of non-zero winding number. It remains to construct a homotopy of γ_0 to the horizontal axis.

Consider the tangent angle $\alpha(t)$ of the curve γ_0; this angle is zero for $|t| >> 1$. Consider the family of functions $s\alpha(t),\ 0 \le s \le 1$. Let γ_s be the immersed curve whose tangent angle is given by this function. This curve is horizontal off a compact, and its left part coincides with the x-axis but it may not be true for the right part. However one can easily adjust γ_s so that it becomes a long curve, and, moreover, do this inside a fixed disc, independently of s.

Remark. Since the winding number is invariant under regular homotopy, one cannot turn an immersed curve with a non-zero winding number inside out. Surprisingly, this can be done with a 2-sphere (there is an interesting movie available about it).

9.4. **Linking number**.

It is intuitively clear that a pair of embedded circles in space may be linked or not. How does one measure such a linking? Let γ and δ be two oriented disjoint curves, and let points x and y traverse the curves. Then the vector xy can be normalized to a unit one, and this gives a map from $\mathbf{T}^2$ to $\mathbf{S}^2$. The degree of this map is called the *linking number* of γ and δ and denoted by $lk\ (\gamma, \delta)$. The linking number does not change under homotopies of the curves as long as they remain disjoint. If there exists a homotopy that takes the curves into two different halfspaces then the linking number is zero (why?)

One can compute the linking number from the projections of the curves on the (horizontal) plane. Choosing the north pole of the unit sphere as a regular value, it follows that the linking number is the algebraic number of those intersection points of the projections of the curves at which γ goes under δ. There are two types of such intersections, and they contribute with the opposite signs.

Corollary. *The linking number changes sign if the orientation of either curve is reversed, and remains the same if both are reversed.*

Remarks. (i) Two circles may be linked even if their linking number vanishes – recall a figure on the blackboard.

(ii) A generic plane projection of a knot is a generic immersed plane curve with the additional information on over/undercrossing, that is, there are no triple points or selftangencies. It is natural to consider knots up to isotopies. Two knots are isotopic if and only if their generic projections are related by a sequence of the Rademeister moves R1, R2, R3: the kink, the selftangency and the triple point moves.

10. Vector fields and Euler characteristic

10.1. **Index of a vector field at a point**.

Recall that a vector field v on a manifold $M \subset \mathbf{R}^n$ is a smooth map $v : M \to \mathbf{R}^n$ such that for every $x \in M$ the vector $v(x) \in T_xM$. Let x be an isolated zero of v. We will define an integer-valued *index* $i(x)$ that will be also sometimes denoted by $i_v(x)$.

First, assume that M is an open domain in $\mathbf{R}^n$. Then $v(t)/|v(t)|$ is a map of a small sphere around x to the unit sphere, and the index $i(x)$ is defined as the degree of this map. This definition can be also applied to a point x such that $v(x) \ne 0$, and the result will be $i(x) = 0$ (why?)

Exercise. Let $v(z) = z^n$; this gives a vector field in the plane with an isolated zero at the origin. The index of this field is n.

To define the index of a vector field on a manifold we consider a coordinate system near a zero of the vector field and apply the previous definition. One must check that this definition is correct, i.e., does not depend on the choice of a coordinate system. This follows from the next lemma.

Lemma. *The index of an isolated zero of a vector field in an open domain in Euclidean space does not change under diffeomorhisms of the domain.*

Proof. Let f be a diffeomorphism, $x' = f(x)$ and $v' = df \circ v \circ f^{-1}$. The statement is that $i_{v'}(x') = i_v(x)$.

We may assume that $x = x' = 0$; consider a convex neighborhood U of the origin. First, let f be orientation preserving. Then consider the family of maps $f_t(y) = f(ty)/t$ for $0 < t \le 1$; define: $f_0(y) = d_0 f(y)$ (this makes sense since $df(y) = \lim_{t\to 0} f(ty)/t$). We claim that this is a smooth homotopy between f and df. Indeed, if $f(y) = x_1 g_1(y) + ... + x_n g_n(y)$ then $f_t(y) = x_1 g_1(ty) + ... + x_n g_n(ty)$, and the claim follows. Moreover, the linear map df is isotopic to the identity.

Thus we have a family of vector fields $v_t = df_t \circ v \circ f_t^{-1}$ with $v_0 = v$ and $v_1 = v'$, and each has the origin as an isolated singularity. Degree of a map does not change under a homotopy, therefore the indices of v and v' are equal.

It remains to consider an orientation reversing f. Such an f can be represented as σg where g is orientation preserving and $\sigma : (x_1, ..., x_n) \to (-x_1, x_2, ..., x_n)$. Then it suffices to prove that $i_{v'}(0) = 1$ where $v' = d\sigma \circ v \circ \sigma^{-1}$. Indeed, $d\sigma = \sigma$ since σ is a linear map, and deg $\sigma = -1$. Thus $i_{v'}(0) = (-1)\ i_v(0)\ (-1) = i_v(0)$.

10.2 **Poincare-Hopf theorem**.

Let M be a compact manifold and v a vector field on M with isolated zeroes. If M has a boundary then we assume that v has the outward direction at each point of the boundary.

Theorem (Poincare-Hopf). *The sum $\sum i_v(x)$ does not depend on the field v (and equals the Euler characteristic $\chi(M)$ – a remark for those who know what Euler characteristic is).*

Example. Consider an even-dimensional sphere, and let v be the vector field "from North to South pole". Then v has two zeroes, the poles, and the indices are equal to 1 (why?) It follows that for every tangent vector field on this sphere the sum of indices of zeroes equals 2; in particular each vector field has at lest one zero.

Start the proof with the simplest case: $M^n \subset \mathbf{R}^n$ is a compact domain in Euclidean space. Consider the (normal) Gauss map $g : \partial M \to bfS^{n-1}$ that assigns the outward unit normal vector to every point of the boundary of M.

The Theorem is a consequence of the next lemma.

Lemma (Hopf). *If v is a vector field in M that has the outward direction at every boundary point (and, in particular, does not vanish on ∂M) then $\sum i_v(x)$ equals the degree of the Gauss map g.*

Proof. Delete a small disc around each zero of v. One obtains a new manifold M' with boundary. Consider the map $v/|v|$ of $\partial M'$ to the unit sphere. This map extends to M', therefore the degree of this map equals zero (Lemma 8.2). That is, the sum of degrees of this map on the small spheres around the zeroes of v plus the degree on ∂M is zero. The former equals $-\sum i_v(x)$ (minus since each small sphere has the "wrong" orientation); the latter equals deg g since v is homotopic to the unit outward normal field. Lemma is proved.

10.3. **Infinitesimal computation of the index**.

The index of an isolated zero of a vector field v at point x cab be found from the partial derivatives of v at x. Consider first the case of a vector field in an open domain $U \subset \mathbf{R}^n$; thus v is a map $U \to \mathbf{R}^n$. A zero is called nondegenerate if the differential $d_x v$ is a nondegenerate linear map.

Lemma. *One has:* $i_v(x) = \pm 1$ *depending on the sign of the determinant of* $d_x v$.

Proof. Since $d_x v$ is nondegenerate, v is a diffeomorphism in a sufficiently small neighborhood of x. Then, as in Lemma 10.1, v can be replaced by its differential, and then by $\pm id$, depending on the sign of det $d_x v$. Lemma is proved.

Next, we extend the above lemma to the case of a general smooth manifold $M^m \subset \mathbf{R}^n$. A vector field v on M can be considered as a map $v : M \to \mathbf{R}^n$, and its differential is a linear map $d_x v : T_x M \to \mathbf{R}^n$.

Lemma. *Let* $v(x) = 0$. *Then* Im $(d_x v) = T_x M$. *Let D be the determinant of the linear map* $d_x v : T_x M \to T_x M$. *Then* $i_v(x) = \pm 1$ *depending on the sign of* D.

Proof. Consider a parameterization $h : U \to M$ of a neighborhood of point x. Let $e_i, \ i = 1, ..., m$ be the basic vectors in $\mathbf{R}^m$; then $t_i = dh_u(e_i) = \partial h / \partial u_i$ is a basis of the tangent space $T_{h(u)} M$. We want to find the image of $t_i(u)$ under the linear map $d_{h(u)} v$. One has:

$$d_{h(u)} v(t_i) = d_u(vh)(e_i) = \partial v(h(u)) / \partial u_i.$$

We can write: $v = \sum_j v_j t_j$ where v_i are some functions. Therefore

$$\partial v(h(u)) / \partial u_i = \sum_j (\partial v_j / \partial u_i) t_j + \sum_j v_j (\partial t_j / \partial u_i).$$

In a zero of the field v the last sum vanishes, and one concludes:

$$d_x v(t_i) = \sum_j (\partial v_j / \partial u_i) t_j \in T_x M.$$

Thus the linear map $d_x v$ maps $T_x M$ to itself and the determinant of this map equals that of the matrix $(\partial v_j / \partial u_i)$. The previous lemma identifies it with the index of the zero of the vector field in U that is diffeomorphic to v under h, and we are done.

10.4. **End of proof of the Poincare-Hopf theorem**.

Now we consider a closed manifold $M^n \subset \mathbf{R}^k$, and let v be a tangent vector field on M with nondegenerte zeroes (in the sense of the previous section). Fix a sufficiently small ϵ, and let N_ϵ be the set of points at distance at least ϵ from M. It is clear that if ϵ is small enough then N_ϵ is a manifold with boundary. The proof of the Poincare-Hopf theorem follows from the next lemma.

Lemma. *The sum of indices* $\sum i_v(x)$ *equals the degree of the normal Gauss map* $\partial N_\epsilon \to \mathbf{S}^{k-1}$.

Proof. For every point $x \in N_\epsilon$ there is a unique point $r(x) \in M$ closest to x (the foot of the perpendicular from x to M). Extend the vector field v to a vector field w on N_ϵ by the formula:

$$w(x) = v(r(x)) + (x - r(x)).$$

Note that w coincides with v on M. We claim that w has the outward direction along the boundary ∂N_ϵ. Indeed, ∂N_ϵ is a level hypersurface of the function $\phi(x) = |x - r(x)|^2$. Then grad $\phi = 2(x - r(x))$. Since gradient is perpendicular to the level hypersurface, the vector $x - r(x)$ is perpendicular to ∂N_ϵ at $x \in \partial N_\epsilon$. Thus the claim is equivalent to the inequality: $w(x) \cdot (x - r(x)) > 0$. The left-hand side is $|x - r(x)|^2$ since $x - r(x)$ is perpendicular to $v(r(x))$, and the claim follows.

According to Lemma 10.2, the sum of indices of the field w on N_ϵ equals the degree of the Gauss map. To finish the proof we need to identify zeroes of w with those of v.

Notice that w is the sum of two orthogonal vectors; thus if $w(x) = 0$ then $x \in M$ and $v(x) = 0$. It remains to see that the indices of w and v at x are equal. This follows from the local computations in the previous section. Indeed, $d_x w(t) = d_x v(t)$ for $t \in T_x M$, and $d_x w(t) = t$ for t perpendicular to $T_x M$. Therefore the determinant of the linear map $d_x w$ equals that of $d_x v$, and the lemma is proved.

One argues similarly in the case when M has a boundary; we will not dwell on that.

10.5. **Vector fields and triangulations**

In this section we will relate Euler characteristic $\chi(M)$ defined as the sum of indices of zeroes of a vector field with a more familiar combinatorial notion (familiar to those who had algebraic topology).

Let M^n be a closed manifold. Let M be *triangulated*, that is, partitioned into n-dimensional simplices (topological images of linear simplices) in such a way that adjacent simplices have a whole face in common. Denote by f_i the number of i-dimensional simplices in the triangulation.

Theorem. $\chi(M) = \sum(-1)^i f_i$.

Proof. We will construct a vector field on M that has a zero at the center of each face of the triangulation; moreover the index of the zero at the center of an i-dimensional face equals $(-1)^i$. Then the result will follow from the Poincare-Hopf theorem.

The construction is inductive. If the field v is already constructed on the boundary of an i-dimensional simplex (that consists of j-dimensional simplices with $j < i$) then we extend v smoothly inside the simplex so that the center of the simplex is an attracting zero (recall the picture on the blackboard). We claim that if x is the center of an i-dimensional face then $i_v(x) = (-1)^i$. Indeed, the linearization of v at x is a diagonal matrix with $n - i$ ones and i minus ones on the diagonal. Hence its determinant is $(-1)^i$, and we are done by a lemma in 10.3.

The above theorem, combined with the Poincare-Hopf theorem, implies that the alternating sum $\sum(-1)^i f_i$ does not depend on the choice of the triangulation, but only on the topology of M.

11. Examples of manifolds

So far, we did not consider any interesting examples of manifolds. The goal of this section is to come up with this shortcoming.

11.1. **Projective spaces**.

Let V be an $n+1$-dimensional vector space; its *projectivization* $P(V)$ is the set of 1-dimensional subspaces (lines through the origin) in V. The space V may be real or complex; the projectivization is called the real or complex projective space, respectively, and denoted by $\mathbf{RP}^n$ or $\mathbf{CP}^n$. We will be mostly speaking about the real case, indicating the differences with the complex one when appropriate (it is your exercise to prove everything in the complex case).

Remark. A similar operation of *spherization* is more familiar to you: $S(V)$ consists of oriented 1-dimensional subspaces (rays through the origin) in V and coincides with n-dimensional sphere.

The previous remark implies that $\mathbf{S}^n$ is a 2-sheeted over of $\mathbf{RP}^n$, namely, that the latter is obtained from the former by identifying pairs of antipodal points. Equivalently, $\mathbf{RP}^n$ is obtained from $V - O$ by identifying proportional vectors: $v \sim tv,\ t \in \mathbf{R},\ t \neq 0$.

The structure of a smooth manifold on $\mathbf{RP}^n$ is defined as follows. Choose coordinates $x_0, ..., x_n$ in V. Then a point of $\mathbf{RP}^n$ is determined by an $(n+1)$-tuple $(x_0, ..., x_n)$ up to a non-zero factor: $(x_0, ..., x_n) \sim t(x_0, ..., x_n)$. Cover $\mathbf{RP}^n$ by $n+1$ open sets $U_i,\ i = 0, ..., n$ where U_i is determined by the condition: $x_i \neq 0$. Then each U_i is in one-to-one correspondence with $\mathbf{R}^n$, namely,

$$f_i : (x_0, ..., x_n) \to \left(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}\right), \quad i \text{ is omitted.}$$

Thus we have coordinate systems on $\mathbf{RP}^n$ that determine the desired structure of a smooth manifold.

Exercise. Compute the transition functions $f_i f_j^{-1}$.

One may wonder how to embed $\mathbf{RP}^n$ in a linear space. The answer is given explicitly by the next lemma whose proof is an exercise.

Lemma. *The formula*

$$(x_0, ..., x_n) \to \frac{\{(x_p x_q)\}}{(\sum_{i \le j} x_i^2 x_j^2)^{1/2}}, \quad 1 \le p, q \le n+1$$

defines an embedding of $\mathbf{RP}^n$ *to* $\mathbf{R}^{(n+1)(n+2)/2}$.

Examples. $\mathbf{RP}^1$ is diffeomorphic to $\mathbf{S}^1$. Indeed, $\mathbf{RP}^1$ is obtained from $\mathbf{R} = U_0$ by adding a "point at infinity", the equivalence class of $(0, 1)$. For the same reason, $\mathbf{CP}^1 = \mathbf{S}^2$, the Riemann sphere.

11.2. **Classical surfaces**.

Unlike higher dimensions, in dimension 2 a complete classification of closed manifolds is available (we formulate it without proof). Oriented manifolds are characterized by a single non-negative integer g called genus; M_g is a sphere with g handles. In particular, M_0 is a sphere, and M_1 is a torus. An alternative description is that M_g is a connected sum of g tori: $M_g = T^2\#...\#T^2$. The surface M_g can be obtained from a $4g$-gon by pasting its sides pairwise – recall the picture on the blackboard. It follows that $\chi(M_g) = 2 - 2g$. The same formula follows from the fact that $\chi(T^2) = 0$ and the next additivity property of Euler characteristic.

Lemma.

$$\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N);$$

and, in particular,

$$\chi(M^2\#N^2) = \chi(M) + \chi(N) - 2.$$

Proof. Suppose that M and N are triangulated so that these triangulations give the same triangulation of their intersection. Then the first formula is just the inclusion-exclusion formula for the number of simplices in the triangulations.

It follows that $\chi(M^2 - D^2) = \chi(M^2) - 1$ and likewise for N^2. Since $\chi(\mathbf{S}^1) = 0$, the second formula follows too.

Similarly, non-oriented closed manifolds are characterized by a positive integer h; the manifold K_h is the connected sum of h copies of the projective plane. Since $\chi(\mathbf{RP}^2) = 1$ (why?), one concludes from the above lemma that $\chi(K_h) = 2 - h$. The surface K_h can also be obtained from a $2h$-gon by pasting its sides pairwise – recall the picture on the blackboard, and this gives another computation of its Euler characteristic.

The connected sum of two projective planes K_2 is called the Klein bottle. A curious property of the connected summation of surfaces is as follows.

Lemma. $\mathbf{RP}^2\#K_2 = \mathbf{RP}^2\#T^2$.

Proof. Delete a disc from the projective plane: what remains is the Moebius band – recall a picture. Attach a handle to the Moebius band and move one of its feet around: what you have now is a connected sum of the Moebius band and the Klein bottle. This finishes the proof.

To summarize, closed 2-dimensional manifolds are characterized by two things: their orientability and Euler characteristic.

11.3. **Lie groups**.

A *Lie group* is a smooth manifold G which is also a group, and the two structures agree in the sense that the inversion and multiplication maps $g \to g^{-1}$ and $(g_1, g_2) \to g_1g_2$ are smooth.

The first example is $\mathbf{R}^n$ (the group operation is summation of vector). Another example of a commutative Lie group is n-dimensional torus.

The group $\mathbf{GL}(n, \mathbf{R})$ consists of non-degenerate $n \times n$ matrices; is it clear that this is a Lie group? An important subgroup of $\mathbf{GL}(n, \mathbf{R})$ is $\mathbf{SL}(n, \mathbf{R})$ which consists of matrices with determinant 1.

Lemma. $\mathbf{SL}(n, \mathbf{R})$ *is a smooth manifold.*

Proof. Consider the function $f(A) = \det A$; we want to how that 1 is a regular value of f. Compute the differential of f at matrix $A \in f^{-1}(1)$:

$$d_A(f)(B) = \lim_{t \to 0} (\det(A + tB) - \det(A))/t = \lim_{t \to 0} (\det(E + tA^{-1}B) - 1)/t = \mathrm{tr}(A^{-1}B),$$

where E is the unit matrix. It follows that $d_A(f)$ is non-degenerate: e.g., $d_A(f)(A) \neq 0$.

Corollary. *The tangent space to* $\mathbf{SL}(n, \mathbf{R})$ *at the unit matrix* E *consists of traceless matrices, i.e. matrices with zero trace.*

Two other important groups are $\mathbf{O}(n)$ and $\mathbf{U}(n)$ that consist, respectively, of the linear transformations of n-dimensional real or complex space that preserve scalar product or Hermitian product. In down-to-earth terms, these groups consists of $n \times n$ real or complex matrices A satisfying $A^*A = E$; here A^* is the transpose matrix in the real case and the transpose complex conjugate matrix in the complex case. The subgroups $\mathbf{SO}(n) \subset \mathbf{O}(n)$ and $\mathbf{SU}(n) \subset \mathbf{U}(n)$ consist of the matrices with the unit determinants.

Example. $\mathbf{SO}(2)$ is diffeomorphic to the circle, and so is $\mathbf{U}(1)$. The group $\mathbf{SU}(2)$ is diffeomorphic to $\mathbf{S}^3$ (why?)

Note the next curious fact.

Proposition. $\mathbf{SO}(3)$ *is diffeomorphic to* $\mathbf{RP}^3$.

Proof. We claim that every transformation from $\mathbf{SO}(3)$ is a rotation about some axis through some angle. Indeed, let $A \in \mathbf{SO}(3)$. It suffices to show that A has a unit eigen-value; then A is a rotation about the respective eigen-space. Consider the eigen problem: $\det\ (A - \lambda E) = 0$. This cubic equation has a real root that must be ± 1. If it is 1 we are done. If it is -1 then A preserves the plane, perpendicular to the eigen-direction, and is a reflection in this plane; then the third eigen-value of A is 1.

We may consider only rotations through the angles from 0 to π; the direction of the rotation gives the axis an orientation (by the left-hand rule). Encode such a rotation by the vector whose direction is that of the axis and whose magnitude is the angle of rotation. We obtain a ball of radius π, but the points on its boundary should be identified with the antipodal point: the rotations through π and $-\pi$ coincide. The result is the real projective space, and we are done.

11.4. **Homogeneous spaces**.

We say that a Lie group G acts on a manifold M if G is a subgroup of the group of diffeomorphisms of M. More formally, we have a map $\phi : G \times M \to M$ and $\phi(g, x)$ is the image of $x \in M$ under $g \in G$. We write $g(x)$ instead of $\phi(g, x)$; the natural equality holds: $(g_1 g_2)(x) = g_1(g_2(x))$. In particular, the unit element of the group is the identity map of M. The defined action is also called *left action*; we will not dwell on what right action is.

Example. An action of a group G on a linear space by linear transformations is called a *linear representation* of G.

An action is called *transitive* if for every two points $x, y \in M$ there exists $g \in G$ such that $g(x) = y$. In other words, the G-orbit of every point is all of M. A manifold M

is called a *homogeneous space* of a group G if G acts transitively on M. The *isotropy subgroup* $G_x \subset G$ of a point $x \in M$ consists of those elements of G that leave x fixed.

Example. A group G acts on itself by left multiplication; this makes G into a homogeneous space.

Exercises. (i) If M is a homogeneous space then the isotropy subgroups G_x and G_y are isomorphic for every two points $x, y \in M$.

(ii) M is in one-to-one correspondence with G/H where H is the isotropy subgroup.

Many important manifolds are homogeneous spaces; below we consider a few examples.

Example. Consider the natural action of the group $\mathbf{SO}(n)$ on the unit sphere $\mathbf{S}^{n-1}$. The isotropy subgroup of a point is $\mathbf{SO}(n-1)$. Thus $\mathbf{S}^{n-1} = \mathbf{SO}(n)/\mathbf{SO}(n-1)$.

Stiffel manifold $V_{n,k}$ consists of k-tuples of orthonormal vectors in $\mathbf{R}^n$. In particular, $V_{n,1} = \mathbf{S}^{n-1}$ and $V_{n,n} = \mathbf{O}(n)$ (why?). Stiffel manifold is a homogeneous space of the group $\mathbf{O}(n)$ that acts as follows: $(e_1, ..., e_k) \to (Ae_1, ..., Ae_k),\ A \in \mathbf{O}(n)$; this action is transitive (why?) The isotropy subgroup of a point $(e_1, ..., e_k)$ consists of the orthogonal transformations that preserve each e_i; these are orthogonal transformations of the space, orthogonal to Span $(e_1, ..., e_k)$, that is, the group $\mathbf{O}(n-k)$. Therefore $V_{n,k} = \mathbf{O}(n)/\mathbf{O}(n-k)$.

Exercise. Show that $V_{n,n-1} = \mathbf{SO}(n)$.

Lemma. *$V_{n,k}$ is a $nk - k(k+1)/2$-dimensional manifold.*

Proof. Stiffel manifold can be considered as a subset in $\mathbf{R}^{nk}$ as follows. Choose some orthonormal basis in $\mathbf{R}^n$ and let $e_i = (x_{i1}, ..., x_{in})$. The numbers x_{ij} are coordinates in $\mathbf{R}^{nk}$, and we have realized $V_{n,k}$ as a subset in $\mathbf{R}^{nk}$. The coordinates x_{ij} are not independent. Namely, since e_i are orthonormal, one has $k(k+1)/2$ equations:

$$\sum_s x_{is} x_{js} = \delta_{ij}.$$

Compute the tangent space to $V_{n,k} \subset \mathbf{R}^{nk}$ at a convenient point X whose coordinates satisfy: $x_{ij} = \delta_{ij}$. Let $X(t) = \{x_{ij}(t)\}$ be a curve on $V_{n,k}$ such that $X(0) = X$. Then

$$\sum_s x_{is}(t) x_{js}(t) = \delta_{ij}; \quad x_{ij}(0) = \delta_{ij}.$$

Denote by v_{ij} the velocity vector dx_{ij}/dt at $t = 0$. Differentiating, one finds: $v_{ij} + v_{ji} = 0,\ i, j = 1, ..., k$. Thus the tangent space in question consists of the vectors v_{ij} with $v_{ij} = -v_{ji},\ i, j = 1, ..., k$. The dimension of this space is as claimed.

Grassmann manifold, or simply, Grassmannian, $G_{n,k}$ is the space of k-dimensional linear subspaces in n-dimensional linear space (real, default, but one can also consider the complex case). In particular, $G_{n,1} = \mathbf{RP}^{n-1}$, and $G_{n,n-1} = \mathbf{RP}^{n-1}$ too (why?) Grassmann manifold is a homogeneous space of the group $\mathbf{O}(n)$ that naturally acts on $\mathbf{R}^n$ and, therefore, on the Grassmannian. The isotropy subgroup of a point is $\mathbf{O}(k) \times \mathbf{O}(n-k)$, thus $G_{n,k} = \mathbf{O}(n)/\mathbf{O}(k) \times \mathbf{O}(n-k)$.

Exercise. Show that $G_{n,k} = G_{n,n-k}$.

We finish by proving that Grassmannian is a smooth manifold and computing its dimension.

Lemma. *$G_{n,k}$ is a $k(n-k)$-dimensional manifold.*

Proof. Consider a point of $G_{n,k}$, that is a k-dimensional subspace V of $\mathbf{R}^n$. Let U be the orthogonal space. Then every k-dimensional subspace, close to V, is the graph of a linear map from V to U. Thus a neighborhood of V is identifies with the space of linear maps from k-dimensional to $n-k$-dimensional space, that is, with the space of $k \times (n-k)$ matrices. We have constructed a parameterization of a neighborhood of V; thus dim $G_{n,k} = k(n-k)$.

Problems

I.

1. Let M be a smooth manifold. Prove that TM (the set of tangent vectors) is also a smooth manifold. Given a smooth map $f: M \to N$, prove that $df: TM \to TN$ is also smooth.

2. Prove that the set of a). nondegenerate $n \times n$ matrices; b). $n \times n$ matrices whose determinant equals 1 is a smooth manifold. Prove that the map $A \to A^{-1}$ is smooth in both cases.

3. Find the singular points and singular values of the map from the plane to itself given, in complex notation, by $z \to z^3 - 3\bar{z}$.

4. Prove that $\mathbf{S}^n \times \mathbf{S}^k$ can be embedded to $\mathbf{R}^{n+k+1}$.

5. Let M^m be a compact manifold (without boundary). Prove that M cannot be immersed to $\mathbf{R}^m$.

II

1. Prove that every map $M^m \to \mathbf{S}^n$ with $m < n$ is homotopic to a constant map.

2. Let M and N be oriented manifolds. Consider the map $\tau : M \times N \to N \times M$ given by $\tau(x, y) = (y, x)$. When is it orientation-preserving?

3. Consider a map $f: \mathbf{C} \to \mathbf{C}$ given by (i) a complex polynomial of degree d; (ii) a rational function P/Q with deg $P = p$, deg $Q = q$. Find deg f.

4. Given a map $f: \mathbf{S}^1 \to \mathbf{S}^1$ which sends antipodal points to antipodal points, prove that $\deg_2 (f) = 1$.

5. Consider a 2×2 matrix A with integer entries. This matrix determines a map f from the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ to itself: if $v \in \mathbf{R}^2$ then $f(v) = Av \bmod \mathbf{Z}^2$. Find deg f.

6. Let $n = 4k-1$. Construct 3 linearly independent tangent vector fields on $\mathbf{S}^n$ (hint: use quaternions). Prove that $T\mathbf{S}^3$ is trivial.

III

1. Given a closed immersed plane curve, prove that the number of its double points has the parity opposite to its winding number.

2. Assuming Whitney's theorem on immersed plane curves, prove that there are exactly two homotopy classes of closed immersed curves on the 2-sphere.

3. What is the relation between $lk\ (\gamma, \delta)$ and $lk\ (\delta, \gamma)$?

4. Let A be a nondegenerate $n \times n$ matrix. Consider the vector field $x \to Ax$ in $\mathbf{R}^n$. Find $i_A(0)$.

5. For each $n = 1, 0, -1, -2, -3, ...$ construct a function $f(x, y)$ such that the index of the vector field grad f at the origin equals n.

IV

1. Prove that the Euler characteristic of an odd-dimensional closed manifold is equal to zero (hint: use the Poincare-Hopf theorem; consider two fields, v and $-v$).

2. Find $\chi(\mathbf{RP}^n)$.

3. Prove that $\mathbf{RP}^{2n+1}$ is orientable while $\mathbf{RP}^{2n}$ is not. In particular, prove that the projective plane cannot be embedded in 3-space.

4. Prove Lemma 11.1.

5. Prove that $\mathbf{O}(n)$ is a smooth manifold, find its dimension and describe it tangent space at E.

6. Let G^n be a Lie group. Prove that G carries n linearly independent tangent vector fields (hint: choose n linearly independent vectors in the tangent space $T_E G$; extend them to a vector field on G using the diffeomorphisms $f_g : G \to G$ where $f_g(h) = gh$).