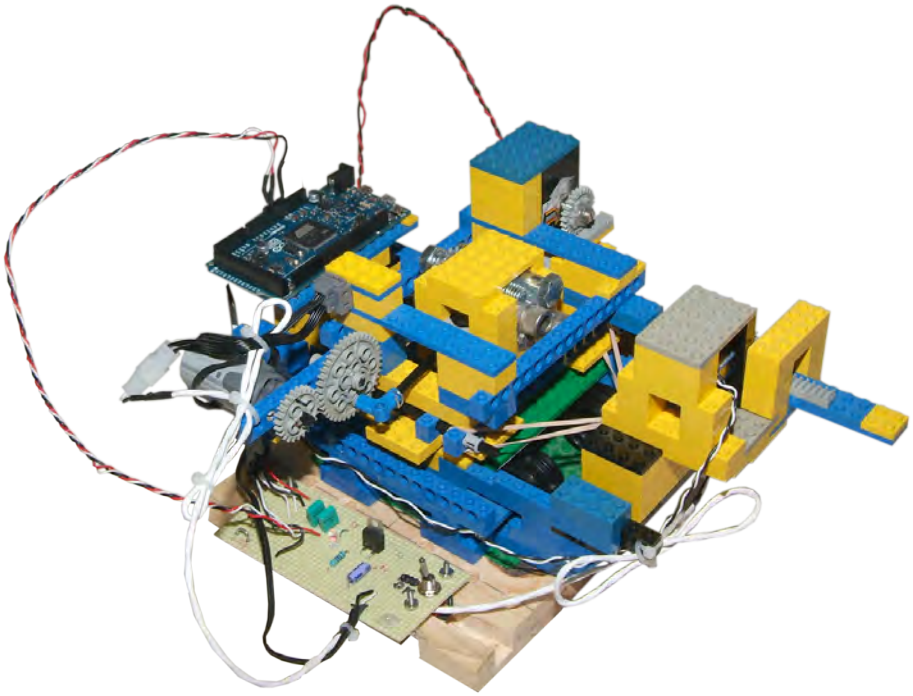
Lego Coriolis Vibrating Gyroscope

by: Scott Gift

2013



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1

Introduction

This document describes the design of a Coriolis vibrating gyroscope created out of Legos. The purpose of this work was two fold: to build a working gyroscope and to understand the math involved with such an endeavor. A gyroscope is a device that outputs (senses) a change in angle. This change in angle over time is referred to as rate. The basic principle of operation of a Coriolis vibrating gyroscope is that an object vibrating along a direction will want to stay vibrating in that same direction even under rotation. When the device undergoes a rotation, some of the vibration is transferred to an orthogonal direction. The amount that is transferred along the orthogonal direction is proportional to the rotational rate. Using this principal, a device can be created to measure the amount of rotational rate being experienced. These type of devices are referred to as gyroscopes.

The Lego gyroscope described within this document was a at-home research project. Initial testing of the Lego gyroscope consisted of mounting it on-top of a cordless drill. Follow-on testing was conducted at the Navigation Research and Development Center (NRDC) in Warminster Pennsylvania. It is from these tests that the data is presented within this document. It is noted that testing was limited and primarily used to verify that the Lego gyroscope worked as a gyroscope. Also, no attempt was made to optimize the design for performance gains.

The Lego gyroscope did measure rate as the next section describes. After this section is detailed analysis on the mathematics used to describe the gyroscope. This is followed by a description of a software simulation designed to verify the system. An overview of the Lego gyroscope is also presented but it is noted that this section is only a overview and not intended to be detailed enough to build a replica. The document finishes with details of the testing effort with analyzed data.

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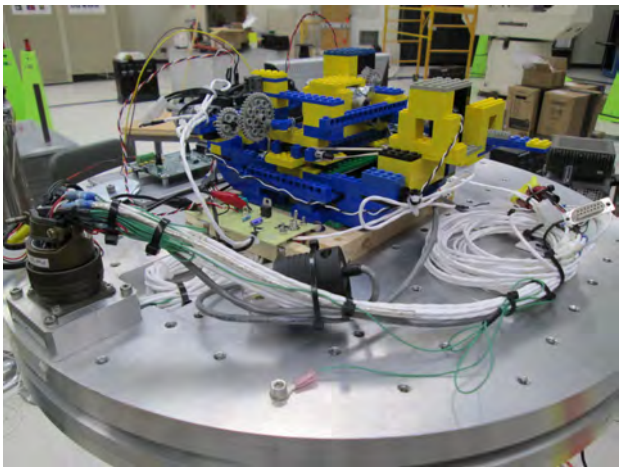
2

Summary and Conclusion

Testing was conducted at the Navigation Research and Development Center (NRDC) in Warminster Pennsylvania on a precision rate table. The purpose of this testing was to prove that the Lego gyroscope did indeed function as a gyroscope. This testing was conducted in May of 2013 over a period of days. Figure 2.1 shows the Lego gyroscope mounted on a precision rate table.



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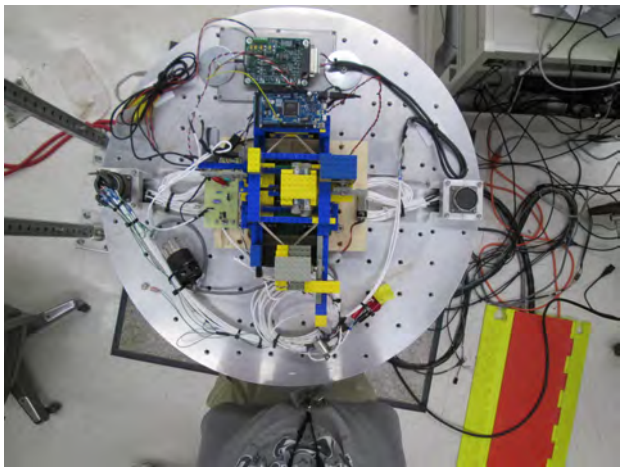

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Figure 2.1: Lego Gyroscope on a Precision Rate Table

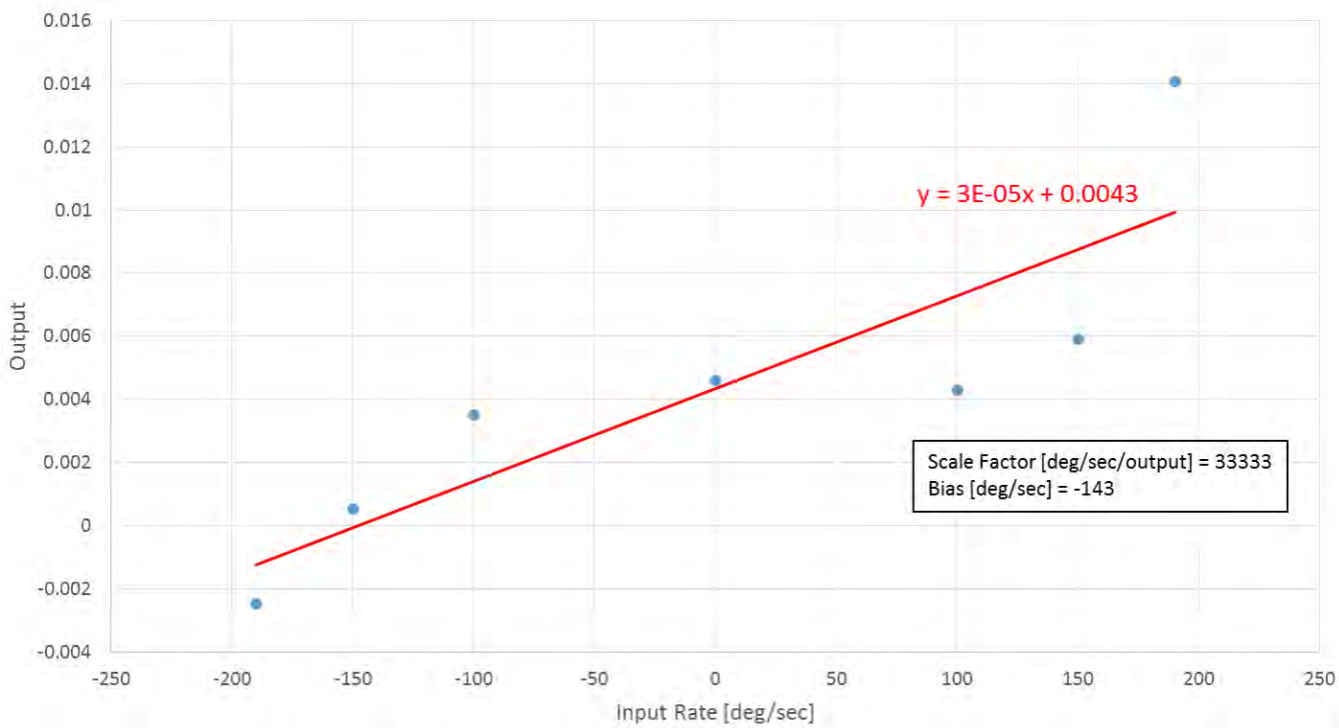

Figure 2.2: Lego Gyroscope Output vs Input Rate

The precision rate table was rotated at ±100 deg/sec, ±150 deg/sec, and ±190 deg/sec. Data was recorded throughout the test effort. From the data collected, a linear fit was applied to the gyro output versus the input rate. This fit is show in Figure 2.2. The Lego gyroscope scale factor was determined to be 33333 deg/sec/output. The Lego gyroscope in-run bias was determined to be -143 deg/sec. The fit also shows a large scale factor non-linearity which is expected because of the Lego design using rubber bands for the springs. The bias-instability was also calculated from another test and determined to be about 11370 deg/hr. Angle Random Walk (ARW) was calculated to be 1063 deg/$\sqrt{\text{hr}}$.

The goals of this project were fulfilled completely. The gyroscope was tested and did measure input rate; the Lego contraption was a gyroscope. The other goal was met through the existence of this document, specifically the analytical section. Going ahead, the author has no plans to improve on the Lego gyroscope design. In fact, many of the parts will probably be reclaimed for other endeavors. Further refining of the device could be accomplished with many simple improvements, but these are left to the reader to further investigate.

3

Theory of Operation Overview

A gyroscope is a device that outputs (senses) a change in angle. This change in angle over time is referred to as rate. The basic principle of operation of a Coriolis vibrating gyroscope is that an object vibrating along a direction will want to stay vibrating in that same direction even under rotation. This is referred to as staying inertial stable. When the device undergoes a rotation, some of the vibration is transferred to an orthogonal axis through the Coriolis effect. The amount that is transferred is proportional to the rotational rate.

More specifically, the Coriolis effect describes mathematically what happens when an object is moving in a rotating frame. The stationary frame of reference can be considered the inertial frame. When an object rotates, its coordinate frame rotates with respect to the inertial frame. This introduces a special force referred to as a fictitious force. This force is what "moves" some of the motion to the other axis.

Figure 3.1 depicts the basic principle of the device. There is a mass that can move in two directions: along the drive axis and the sense axis. An oscillation is driven along the drive axis of the device. This moves the mass back and forth at a specified frequency.

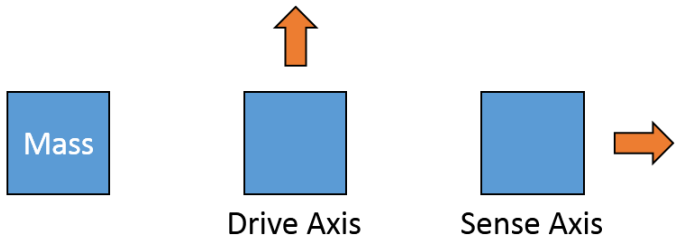

Figure 3.1: Drive and Sense Axes

When the oscillating mass is rotated, some of the oscillation will transfer to the other axis. The amount of oscillation transferred is proportional to the angular rate as shown in figure 3.2.

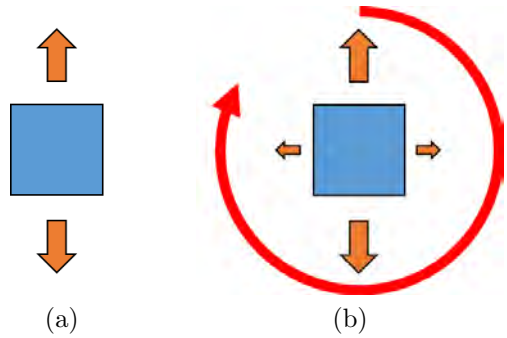

Figure 3.2: (a) Mass oscillating along the drive axis; (b) mass under rotation also oscillates some along the sense axis

For the Lego gyroscope developed, a direct linkage from the motor to the mass was used to create the oscillation along the drive axis. This design is shown in figure 3.3. A Lego motor was connected directly to a Lego gear. One side of a Lego shaft was connected to the edge of the gear and the other side to the mass. In this configuration, the constantly rotating motor was used to make the mass oscillate back and forth.

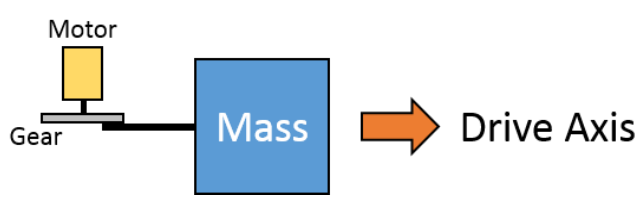

Figure 3.3: Drive Axis Setup

The sense axis was created using rubber bands and placing the entire assembly on wheels to lower friction (damping). The rubber bands acted as springs. To maximize the gyroscope potential, the sense axis was tuned to resonate at the same frequency as the oscillating mass. Figure 3.4 shows a typical mass, spring, damper system. This type of system creates a differential equation. The analysis section of this document describes this in more detail.

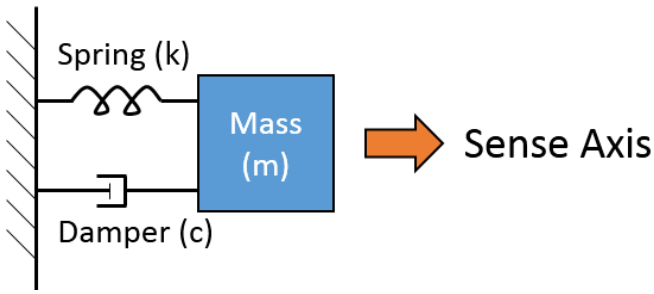

Figure 3.4: Drive Axis Setup

The motor oscillated the mass back and forth at a constant frequency along the drive axis. The sense axis was tuned close to the frequency of the drive axis. Then when a rotation rate was applied, the sense axis also oscillated. The magnitude of oscillation was proportional to the magnitude of the input rate and measured by the supporting electronics.

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4

Detailed Analysis

This chapter describes how the Lego Coriolis vibrating gyroscope operates mathematically. The following is an overview of the system. The next section describes in detail the differential equation used for a typical mass, damper, and spring system with a sinusoidal forcing function. A closed form solution is presented. A derivation of the Coriolis force is presented next. This is followed by a short description on extracting the rate from the measurement.

The mechanical system for the Lego gyroscope is shown in figure 4.1. The mass M_1 is sinusoidally vibrated along the x-axis by a motor. This is considered the drive axis. Masses M_1 and M_2 can freely move along the y-axis. This is considered the sense axis. When a rotation is applied about the z-axis (out of the page), then some of the vibration will be transferred to the y-axis.

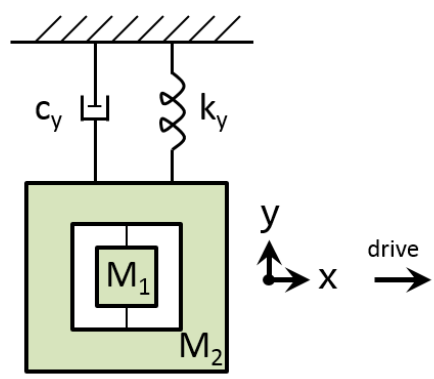


Figure 4.1: Mechanical Model of the System

The motion along the x-axis can be described in the following equations. For this Lego gyroscope, D_x is approximate 20 mm.

$$x = D_x cos(\omega t)$$
$$\dot{x} = -\omega D_x sin(\omega t)$$

The Coriolis force is important to the Lego gyroscope because it is this fictitious-force that allows the sense axis to pickup a portion of the motion along the drive axis when the Lego gyroscope is rotating. The Coriolis force with a constant input rate of Ω_z will be the following.

$$\text{Coriolis force} = -2M_1\dot{x}\Omega_z$$

The equation of motion along the y-axis will be a spring, mass, and damper system with the Coriolis force as shown in equation 4.1.

$$(M_1 + M_2)\,\ddot{y} + c_y\dot{y} + k_y = 2\omega M_1 D_x \Omega_z sin(\omega t) \tag{4.1}$$

The particular solution to the differential is shown in equation 4.2. The next section details the steps to derive the this solution.

$$y_p = 2\omega M_1 D_x \Omega_z \rho_y sin(\omega t - \alpha_y) \tag{4.2}$$

$$\rho_y = \frac{1}{\sqrt{(k_y - (M_1 + M_2)\omega^2)^2 + (c_y\,\omega)^2}}$$
$$\alpha_y = tan^{-1}\left(\frac{c_y\,\omega}{k_y - (M_1 + M_2)\omega^2}\right)$$

The resonant frequency, f_0, of the system is then:

$$\omega_0 = \sqrt{\left(\frac{k_y}{(M_1 + M_2)} - \frac{{c_y}^2}{2(M_1 + M_2)^2}\right)} \tag{4.3}$$

$$f_0 = \frac{\omega_0}{2\pi} \tag{4.4}$$

4.1 Differential Equation of Motion

The mechanical system described in figure 4.1 consists of a mass, a damper, and a spring. The equation that describes this motions is shown in equation 4.5.

$$m\ddot{y} + c\dot{y} + ky = F \tag{4.5}$$

m is the mass, c is the damping coefficient, k is the spring coefficient, and F is the forcing function. Lego gyroscope's motor oscillates the mass at a constant frequency and is simply modeled as $F_0 cos(\omega t + \theta)$. The full differential equation is then shown to be:

$$m\ddot{y} + c\dot{y} + ky = F_0 cos(\omega t + \theta) \tag{4.6}$$

4.1.1 Particular Solution

The particular solution to this problem will have the form:

$$y_p = Acos(\omega t + \theta) + Bsin(\omega t + \theta) \tag{4.7}$$

To solve for A and B, the derivatives of the particular solution (y_p, $\dot{y}_p$, and $\ddot{y}_p$) are calculated and plugged back into the original equation. This then reduces down to equations 4.8 and 4.9 in the form of two equations with two unknowns.

Calculating the derivative of the particular solution yields:

$$\begin{aligned} y_p &= Acos(\omega t + \theta) + Bsin(\omega t + \theta) \\ \dot{y}_p &= -A\omega sin(\omega t + \theta) + B\omega cos(\omega t + \theta) \\ \ddot{y}_p &= -A\omega^2 cos(\omega t + \theta) - B\omega^2 sin(\omega t + \theta) \end{aligned}$$

Substituting the particular solution into equation 4.6 yields:

$$\begin{aligned}
&-mA\omega^2 cos(\alpha) - mB\omega^2 sin(\alpha) - \\
&\quad cA\omega sin(\alpha) + cB\omega cos(\alpha) + \\
&\qquad kAcos(\alpha) + Bksin(\alpha) = F_0 cos(\alpha) \qquad \text{where } \alpha = \omega t + \theta
\end{aligned}$$

After collecting the terms and rearranging the equation becomes:

$$\begin{aligned}
\left(-mA\omega^2 + cB\omega + kA\right) cos(\alpha) &= F_0 cos(\alpha) \\
\left(-mB\omega^2 - cA\omega + kB\right) sin(\alpha) &= 0
\end{aligned}$$

This reduces to:

$$\left(-mA\omega^2 + cB\omega + kA\right) = F_0 \tag{4.8}$$

$$\left(-mB\omega^2 - cA\omega + kB\right) = 0 \tag{4.9}$$

Shown in in matrix form:

$$\begin{bmatrix} (k - m\omega^2) & c\,\omega \\ -c\,\omega & (k - m\omega^2) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \tag{4.10}$$

To get the solution to the coefficients A and B, each side of equation 4.10 is multiplied by the inverse of the model coefficient terms matrix. This changes the equation to:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} (k - m\omega^2) & c\,\omega \\ -c\,\omega & (k - m\omega^2) \end{bmatrix}^{-1} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \tag{4.11}$$

The general form of 2x2 matrix and inverse is:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse of the model coefficient matrix is then:

$$\begin{bmatrix} (k - m\omega^2) & c\,\omega \\ -c\,\omega & (k - m\omega^2) \end{bmatrix}^{-1} = \frac{1}{(k - m\omega^2)(k - m\omega^2) + (c\,\omega)(c\,\omega)} \begin{bmatrix} (k - m\omega^2) & -c\,\omega \\ c\,\omega & (k - m\omega^2) \end{bmatrix}$$

Equation 4.11 has a zero in the matrix to the right which can be used to make the solution simpler. This simplification is:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} d \\ -cF_0 \end{bmatrix}$$

The A and B coefficients are then solved to be:

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\,\omega)^2} \tag{4.12}$$

$$B = \frac{(c\,\omega)F_0}{(k - m\omega^2)^2 + (c\,\omega)^2} \tag{4.13}$$

Notice that in equations 4.12 and 4.13, the A and B coefficients have the same denominator. Also notice that the numerators are terms from the denominator. These terms can then be expressed in terms of a *sin* and *cos* as shown in the triangle diagram, Figure 4.2, and following equations [3]:

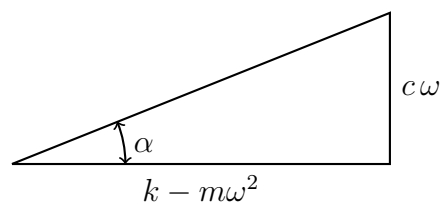


Figure 4.2: Relationship between the A and B coefficients

$$\text{hypotenuse} = \sqrt{(k - m\omega^2)^2 + (c\,\omega)^2}$$

$$cos(\alpha) = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\,\omega)^2}}$$

$$sin(\alpha) = \frac{c\,\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\,\omega)^2}}$$

Using a new term ρ, coefficients A and B can be simply expressed as:

$$\rho = \frac{1}{\sqrt{(k - m\omega^2)^2 + (c\,\omega)^2}}$$

$$A = \rho F_0 cos(\alpha) \tag{4.14}$$

$$B = \rho F_0 sin(\alpha) \tag{4.15}$$

The A and B coefficients, equations 4.14 and 4.15, then get plugged back into the particular solution:

$$\begin{aligned} y_p &= Acos(\omega t + \theta) + Bsin(\omega t + \theta) \\ &= \rho F_0 cos(\alpha) cos(\omega t + \theta) + \rho F_0 sin(\alpha) sin(\omega t + \theta) \end{aligned}$$

Using the trigonometric identity $cos(x_1 - x_2) = cos(x_1)cos(x_2) + sin(x_1)sin(x_2)$, the analytical solution to the differential problem is shown to be:

$$y_p = \rho F_0 cos(\omega t + \theta - \alpha) \tag{4.16}$$

$$\rho = \frac{1}{\sqrt{(k - m\omega^2)^2 + (c\,\omega)^2}} \tag{4.17}$$

$$\alpha = tan^{-1}\left(\frac{c\,\omega}{k - m\omega^2}\right) \tag{4.18}$$

4.1.2 Resonant Frequency

The resonant frequency is the place in hertz that the system will give the maximum excitation. In the case for this given system, it is the place of maximum displacement. The resonant frequency for equations 4.17 and 4.18 can be calculated analytically by finding the minimum of ρ's denominator. This will make the amplitude of the sinusoids the highest. This is accomplished by finding where the denominator's derivative is zero. The solution is to determine at what ω the system resonate. This is referred to as ω_0.

$$
\begin{aligned}
g(\omega_0) &= (k - m{\omega_0}^2)^2 + (c\,\omega_0)^2 \\
&= k^2 - 2km\omega_0^2 + m^2\omega_0^4 + c^2\omega_0^2 \\
\dot{g}(\omega_0) &= -4km\omega_0 + 4m^2{\omega_0}^3 + 2c^2\omega_0
\end{aligned}
$$

$$\text{the minimum is when } \dot{g}(\omega_0) = 0$$

$$
\begin{aligned}
0 &= 4m^2{\omega_0}^3 + \left(2c^2 - 4km\right)\omega_0 \\
0 &= {\omega_0}^3 + \left(\frac{2c^2}{4m^2} - \frac{4km}{4m^2}\right)\omega_0 \\
0 &= {\omega_0}^3 + \left(\frac{c^2}{2m^2} - \frac{k}{m}\right)\omega_0 \\
{\omega_0}^3 &= -\left(\frac{c^2}{2m^2} - \frac{k}{m}\right)\omega_0 \\
{\omega_0}^2 &= \left(\frac{k}{m} - \frac{c^2}{2m^2}\right) \\
\omega_0 &= \sqrt{\left(\frac{k}{m} - \frac{c^2}{2m^2}\right)}
\end{aligned}
$$

Since $\omega_0 = 2\pi f_0$, the resonant frequency in hertz (hz) can be calculated by:

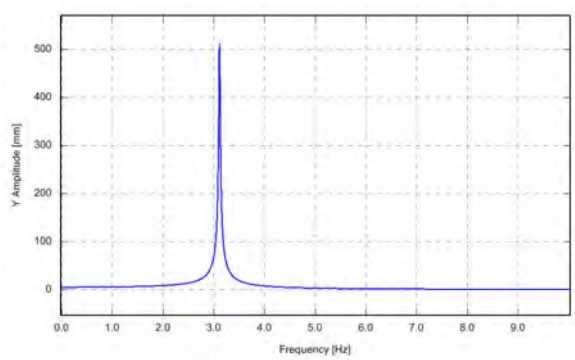
$$f_0 = \left(\frac{1}{2\pi}\right)\sqrt{\left(\frac{k}{m} - \frac{c^2}{2m^2}\right)} \tag{4.19}$$

Table 4.1 shows two cases of the using equation 4.19. Case 1 is more ideal while Case 2 uses the parameters as measured from the Lego gyroscope. As can be seen, the resonant frequency in both cases is similar.

Parameter	Units	Case 1	Case 2
m	kg	0.528	0.528
c	kg/sec	0.1	5.1
k	N/meter	202	202
ω_0	rad/sec	19.5	18.2
f	Hz	3.1	2.9

Table 4.1: Resonant Frequency Parameters

The difference between the two cases is seen in figures 4.3 and 4.4. Case 1 (blue) is narrow with a high magnitude while and Case 2 (red) is wide with a small magnitude. The thin and high resonance plot is the better figure to have for a Coriolis vibrating gyroscope. However, the Lego design inherently had a larger damping (c) term than the ideal case and hence the more broad and lower magnitude shape shown in red.



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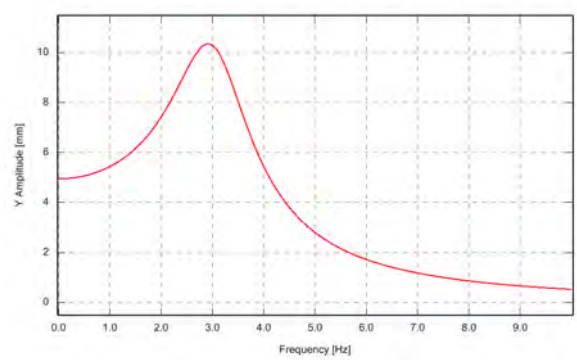

(b)

Figure 4.3: Resonant Frequency Plots of Case 1 (blue) and Case 2 (red)

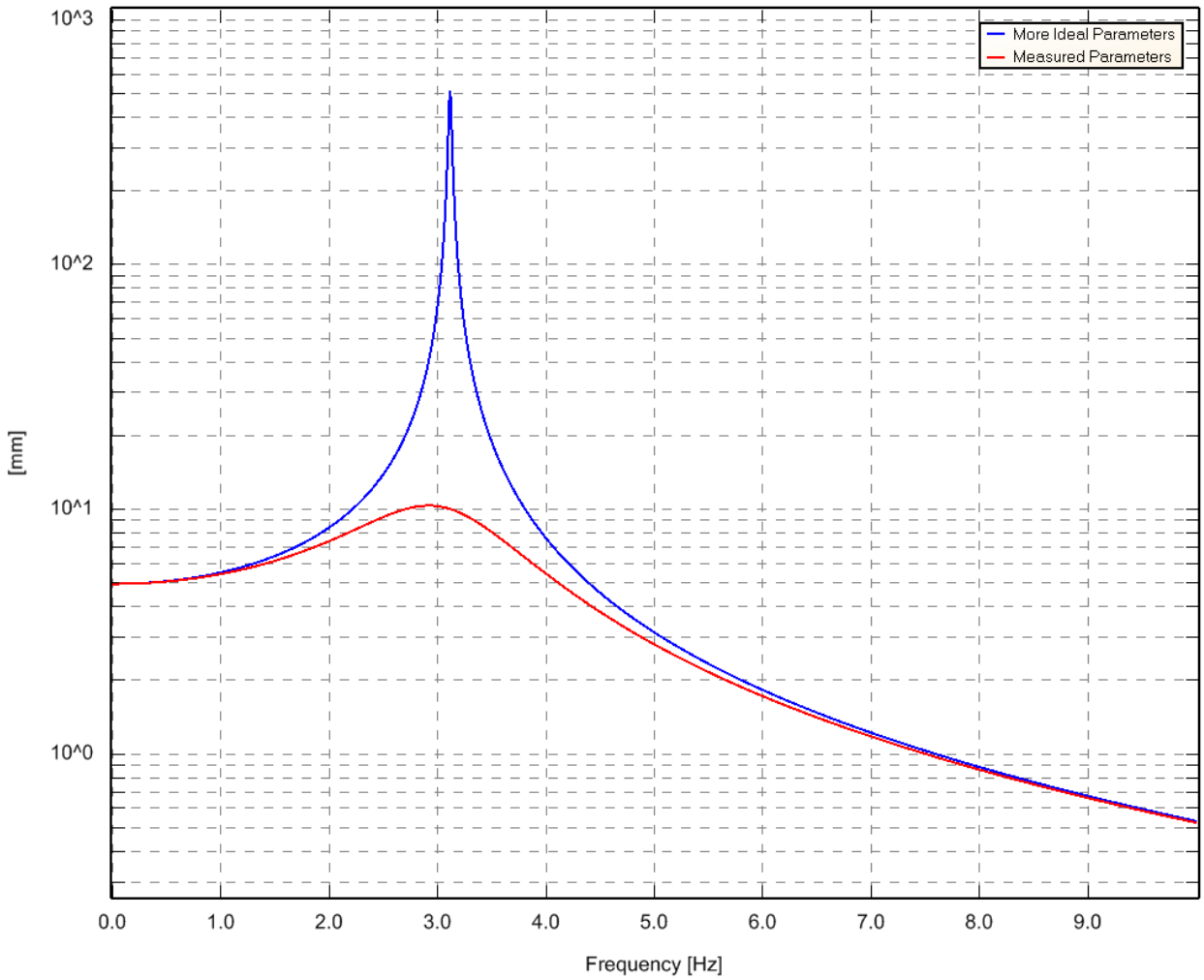

Figure 4.4: Single Resonant Frequency Plot of Case 1 (blue) and Case 2 (red)

4.1.3 Homogeneous Solution

The homogeneous solution to the differential equation along the sense axis is useful in determining the system's parameters. The mass (m) of the system for the Lego gyroscope was determined by measurement with a scale. The spring (k) and damper (c) terms were determined by physically moving the Lego gyroscope's mass to the left or right along the sense axis then letting go as illustrated in figure 4.5. When "let go," the system would oscillate with an exponential decay. This type of response is expected and shown by the solution to the homogeneous equation, figure 4.6.

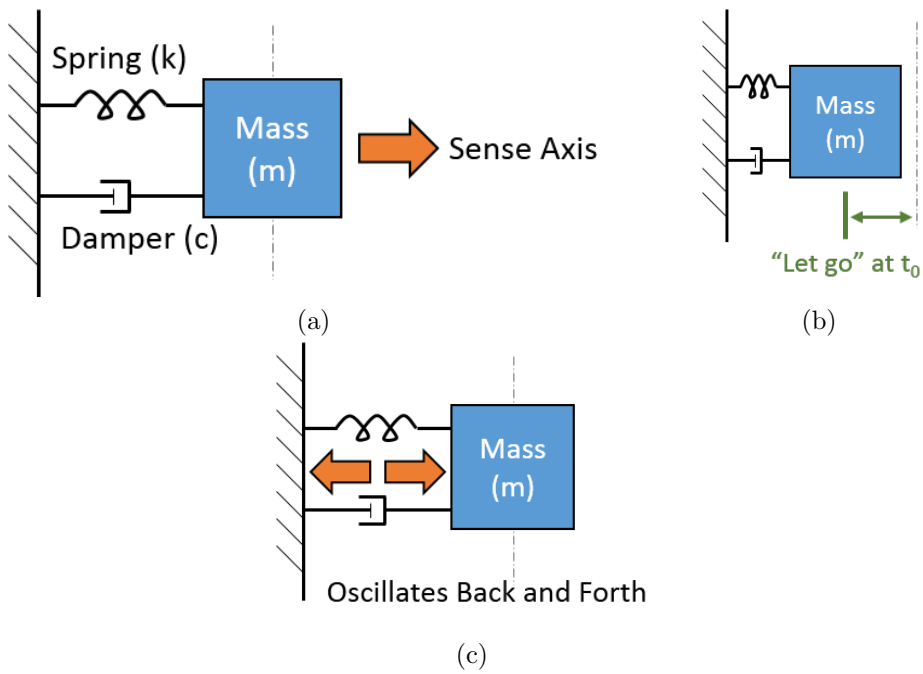

Figure 4.5: (a) Sense axis consists of a mass, spring, and damper; (b) the mass is moved to one side and then "let go"; (c) after the 'let go," the mass oscillates with an exponential decay

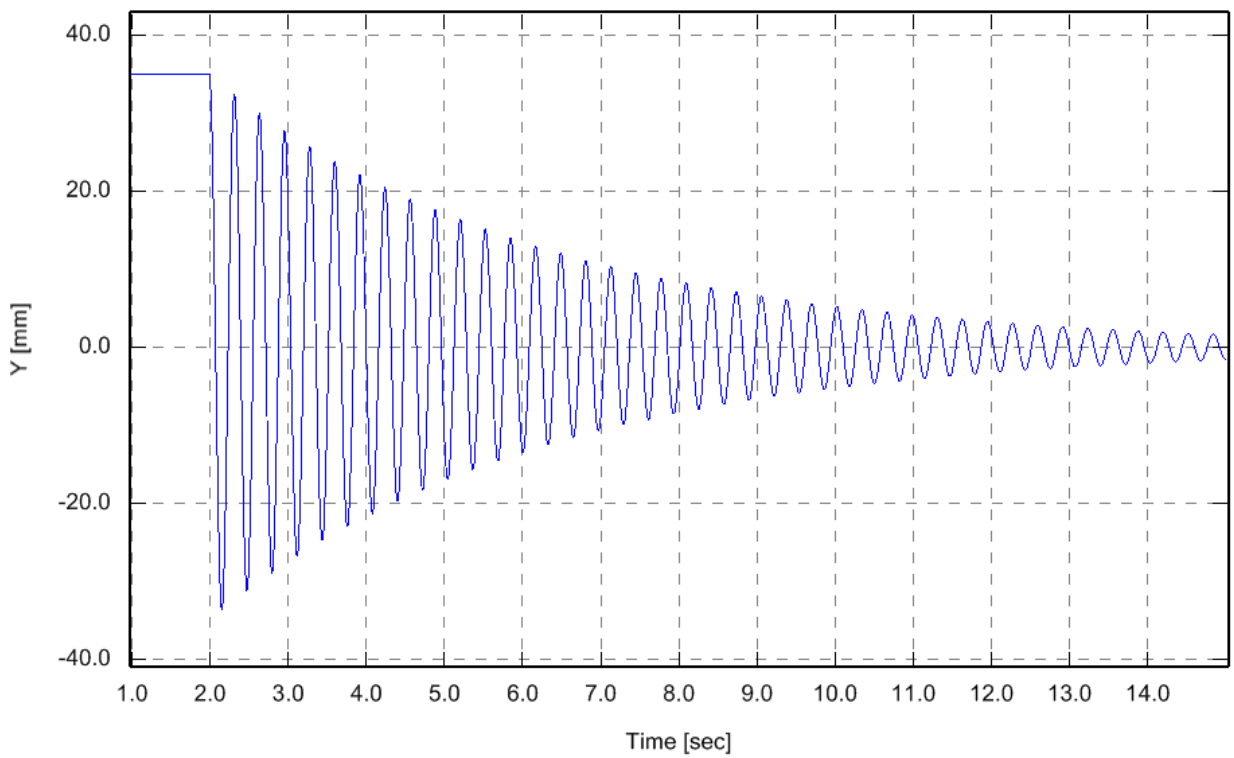

Figure 4.6: Typical response to a homogeneous equation with complex roots; the mass is "let go" at 2 sec

The following equation is the characteristic equation to the homogeneous solution (y_h).

$$mr^2 + cr + k = 0$$

The roots of the characteristic equation are:

$$\{r\} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

There exist three possible solution types to the homogeneous equation. The solutions are depending on the types of the roots: distinct real roots, repeated roots, or complex roots.

Distinct Real Roots ($c^2 > 4mk$)

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}$$

$$r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}$$

$$y_h = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Repeated Roots ($c^2 == 4mk$)

$$r_1, r_2 = \frac{c}{2m}$$

$$y_h = (C_1 + C_2 t)\, e^{\frac{-c}{2m} t}$$

Complex roots ($c^2 < 4mk$)

$$r = \frac{-c}{2m} \pm \frac{\sqrt{4mk - c^2}}{2m} i \tag{4.20}$$

$$y_h = e^{\frac{-c}{2m} t} \left[C_1 cos\left(\frac{\sqrt{4mk - c^2}}{2m} t \right) + C_2 sin\left(\frac{\sqrt{4mk - c^2}}{2m} t \right) \right] \tag{4.21}$$

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4.2 Coriolis Force

This section mathematically derives the fictitious Coriolis force and is based on the paper *Equations of Motion For Rotating Frames* by Nick Saluzzi [1]. The Coriolis force is important to the Lego gyroscope because it is this fictitious-force that allows the sense axis to pickup a portion of the motion along the drive axis when the Lego gyroscope is rotating. This section concludes with the general form for the motion equation with respect to a rotating frame about the z-axis. This is shown in equation 4.22 with a more specific equation shown in equation 4.23.

Figure 4.7 depicts a two dimensional scenario of two independent coordinate frames. **P** is the location of an arbitrary object with mass having three components: x, y, z (note, only two dimensions are shown in the figure). Vector $\mathbf{V}_A$ represents the position of the object in coordinate frame A. Vector $\mathbf{V}_B$ represents the position of the object in coordinate frame B. Vector $\mathbf{V}_{AB}$ represents the position of coordinate B in coordinate frame A.

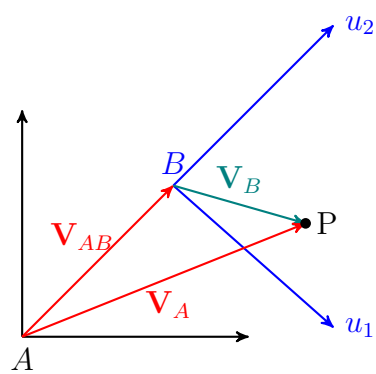

Figure 4.7: 2D representation of two coordinate frames

It can be seen that Vector $\mathbf{V}_A$ is the summation of $\mathbf{V}_{AB}$ and $\mathbf{V}_B$.

$$\mathbf{V}_A = \mathbf{V}_{AB} + \mathbf{V}_B$$

This equation can be represented as the summation of the vector components:

$$\begin{bmatrix} V_{A,x} \\ V_{A,y} \\ V_{A,z} \end{bmatrix} = \begin{bmatrix} V_{AB,x} \\ V_{AB,y} \\ V_{AB,z} \end{bmatrix} + \begin{bmatrix} V_{B,x} \\ V_{B,y} \\ V_{B,z} \end{bmatrix}$$

This can be broken down further:

$$\mathbf{V}_B = V_{B,x}\begin{bmatrix}1\\0\\0\end{bmatrix} + V_{B,y}\begin{bmatrix}0\\1\\0\end{bmatrix} + V_{B,z}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Vector $\mathbf{V}_B$ is can be rewritten as the sum of unit vectors: $\mathbf{u}_x$, $\mathbf{u}_y$, and $\mathbf{u}_z$.

$$\mathbf{u}_x = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathbf{u}_y = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \mathbf{u}_z = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Then this can be put into sigma notation:

$$\begin{aligned}\mathbf{V}_B &= V_{B,x}\mathbf{u}_x + V_{B,y}\mathbf{u}_y + V_{B,z}\mathbf{u}_z \\ &= \sum_k \left(V_{B,k}\mathbf{u}_k\right) \\ &\text{where } k = \{x, y, z\}\end{aligned}$$

Equations are typically written as a summation of forces. A force consists of a mass and an acceleration. This infers that the vector equations need to be expressed in terms of acceleration vectors.

$$\ddot{\mathbf{V}}_A = \frac{d^2}{dt^2}\mathbf{V}_A$$

The derivatives are expressed as:

$$\begin{aligned}\mathbf{V}_A &= \mathbf{V}_{AB} + \sum_k \left(V_{B,k}\mathbf{u}_k\right) \\ \dot{\mathbf{V}}_A &= \dot{\mathbf{V}}_{AB} + \sum_k \left(\dot{\mathbf{V}}_{B,k}\mathbf{u}_k\right) + \sum_k \left(V_{B,k}\dot{\mathbf{u}}_k\right) \\ \ddot{\mathbf{V}}_A &= \ddot{\mathbf{V}}_{AB} + \sum_k \left(\ddot{V}_{B,k}\mathbf{u}_k\right) + \sum_k \left(\dot{V}_{B,k}\dot{\mathbf{u}}_k\right) + \sum_k \left(\dot{V}_{B,k}\dot{\mathbf{u}}_k\right) + \sum_k \left(V_{B,k}\ddot{\mathbf{u}}_k\right) \\ &= \ddot{\mathbf{V}}_{AB} + \ddot{\mathbf{V}}_{B,k} + 2\sum_k \left(\dot{V}_{B,k}\dot{\mathbf{u}}_k\right) + \sum_k \left(V_{B,k}\ddot{\mathbf{u}}_k\right)\end{aligned}$$

For the single axis gyroscope, there is a rotation about the z-axis as specified by $\mathbf{R}_z(\theta)$:

$$\mathbf{R}_z(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This rotation can be applied to the unit vectors:

$$\begin{aligned}
\overline{\mathbf{u}}_x = \mathbf{R}_z(\theta)\mathbf{u}_x &= \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} cos(\theta) \\ sin(\theta) \\ 0 \end{bmatrix} \\
\overline{\mathbf{u}}_y = \mathbf{R}_z(\theta)\mathbf{u}_y &= \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -sin(\theta) \\ cos(\theta) \\ 0 \end{bmatrix} \\
\overline{\mathbf{u}}_z = \mathbf{R}_z(\theta)\mathbf{u}_z &= \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

The first derivatives (rate) of the rotated unit vectors are:

$$\begin{aligned}
\dot{\overline{\mathbf{u}}}_x &= \begin{bmatrix} -\dot{\theta}sin(\theta) \\ \dot{\theta}cos(\theta) \\ 0 \end{bmatrix} \\
\dot{\overline{\mathbf{u}}}_y &= \begin{bmatrix} -\dot{\theta}cos(\theta) \\ -\dot{\theta}sin(\theta) \\ 0 \end{bmatrix} \\
\dot{\overline{\mathbf{u}}}_z &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

The second derivatives (acceleration) of the rotated unit vectors are:

$$\ddot{\overline{\mathbf{u}}}_x = \begin{bmatrix} -\ddot{\theta}sin(\theta) - \dot{\theta}^2cos(\theta) \\ \ddot{\theta}cos(\theta) - \dot{\theta}^2sin(\theta) \\ 0 \end{bmatrix}$$
$$\ddot{\overline{\mathbf{u}}}_y = \begin{bmatrix} -\ddot{\theta}cos(\theta) + \dot{\theta}^2sin(\theta) \\ -\ddot{\theta}sin(\theta) - \dot{\theta}^2cos(\theta) \\ 0 \end{bmatrix}$$
$$\ddot{\overline{\mathbf{u}}}_z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The first derivative can expressed as:

$$\dot{\overline{\mathbf{u}}}_x = \dot{\theta}\overline{\mathbf{u}}_y$$
$$\dot{\overline{\mathbf{u}}}_y = -\dot{\theta}\overline{\mathbf{u}}_x$$
$$\dot{\overline{\mathbf{u}}}_z = [0, 0, 0]^T$$

The second derivative similarly be can expressed as:

$$\ddot{\overline{\mathbf{u}}}_x = \ddot{\theta}\overline{\mathbf{u}}_y - \dot{\theta}^2\overline{\mathbf{u}}_x$$
$$\ddot{\overline{\mathbf{u}}}_y = -\ddot{\theta}\overline{\mathbf{u}}_x - \dot{\theta}^2\overline{\mathbf{u}}_y$$
$$\ddot{\overline{\mathbf{u}}}_z = [0, 0, 0]^T$$

Using the following substitution:

$$\frac{\dot{\overline{\mathbf{u}}}_x}{\dot{\theta}} = \overline{\mathbf{u}}_y$$
$$\frac{-\dot{\overline{\mathbf{u}}}_y}{\dot{\theta}} = -\overline{\mathbf{u}}_x$$

The second derivative can now be expressed as:

$$\begin{aligned}
\ddot{\overline{\mathbf{u}}}_x &= \ddot{\theta}\overline{\mathbf{u}}_y + \dot{\theta}\dot{\overline{\mathbf{u}}}_y \\
\ddot{\overline{\mathbf{u}}}_y &= -\ddot{\theta}\overline{\mathbf{u}}_x - \dot{\theta}\dot{\overline{\mathbf{u}}}_x \\
\ddot{\overline{\mathbf{u}}}_z &= [0, 0, 0]^T
\end{aligned}$$

The rate vector is expressed as:

$$\mathbf{\Omega} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

The cross product of $\mathbf{\Omega}$ and $\overline{\mathbf{u}}_x$ is:

$$\begin{aligned}
\mathbf{\Omega} \times \overline{\mathbf{u}}_x &= \begin{vmatrix} u_x & u_y & u_z \\ 0 & 0 & \dot{\theta} \\ cos(\theta) & sin(\theta) & 0 \end{vmatrix} \\
&= \begin{bmatrix} -\dot{\theta}sin(\theta) \\ \dot{\theta}cos(\theta) \\ 0 \end{bmatrix} \\
&= \dot{\overline{\mathbf{u}}}_x
\end{aligned}$$

Similarly:

$$\begin{aligned}
\mathbf{\Omega} \times \overline{\mathbf{u}}_y &= \begin{vmatrix} u_x & u_y & u_z \\ 0 & 0 & \dot{\theta} \\ -sin(\theta) & cos(\theta) & 0 \end{vmatrix} \\
&= \begin{bmatrix} -\dot{\theta}cos(\theta) \\ -\dot{\theta}sin(\theta) \\ 0 \end{bmatrix} \\
&= \dot{\overline{\mathbf{u}}}_y
\end{aligned}$$

The derivatives of the cross products are:

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{\Omega} \times \overline{\mathbf{u}}_x) &= \dot{\boldsymbol{\Omega}} \times \overline{\mathbf{u}}_x + \boldsymbol{\Omega} \times \dot{\overline{\mathbf{u}}}_x \\ &= \dot{\boldsymbol{\Omega}} \times \overline{\mathbf{u}}_x + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \overline{\mathbf{u}}_x)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{\Omega} \times \overline{\mathbf{u}}_y) &= \dot{\boldsymbol{\Omega}} \times \overline{\mathbf{u}}_y + \boldsymbol{\Omega} \times \dot{\overline{\mathbf{u}}}_y \\ &= \dot{\boldsymbol{\Omega}} \times \overline{\mathbf{u}}_y + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \overline{\mathbf{u}}_y)\end{aligned}$$

$$\dot{\boldsymbol{\Omega}} \times \overline{\mathbf{u}}_x = \begin{vmatrix} u_x & u_y & u_z \\ 0 & 0 & \ddot{\theta} \\ cos(\theta) & sin(\theta) & 0 \end{vmatrix} = \begin{bmatrix} -\ddot{\theta} sin(\theta) \\ \ddot{\theta} cos(\theta) \\ 0 \end{bmatrix} = \ddot{\theta}\overline{\mathbf{u}}_y$$

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \overline{\mathbf{u}}_x) = \boldsymbol{\Omega} \times \begin{bmatrix} -\dot{\theta} sin(\theta) \\ \dot{\theta} cos(\theta) \\ 0 \end{bmatrix} = \begin{vmatrix} u_x & u_y & u_z \\ 0 & 0 & \dot{\theta} \\ -\dot{\theta} sin(\theta) & \dot{\theta} cos(\theta) & 0 \end{vmatrix} = \begin{bmatrix} -\dot{\theta}^2 cos(\theta) \\ -\dot{\theta}^2 sin(\theta) \\ 0 \end{bmatrix} = \dot{\theta}\dot{\overline{\mathbf{u}}}_y$$

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{\Omega} \times \overline{\mathbf{u}}_x) &= \dot{\boldsymbol{\Omega}} \times \overline{\mathbf{u}}_x + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \overline{\mathbf{u}}_x) \\ &= \ddot{\theta}\overline{\mathbf{u}}_y + \dot{\theta}\dot{\overline{\mathbf{u}}}_y \\ &= \ddot{\overline{\mathbf{u}}}_x\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{\Omega} \times \overline{\mathbf{u}}_y) &= \dot{\boldsymbol{\Omega}} \times \overline{\mathbf{u}}_y + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \overline{\mathbf{u}}_y) \\ &= -\ddot{\theta}\overline{\mathbf{u}}_x - \dot{\theta}\dot{\overline{\mathbf{u}}}_x \\ &= \ddot{\overline{\mathbf{u}}}_y\end{aligned}$$

The general form is then:

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{\Omega} \times \overline{\mathbf{u}}_k) &= \dot{\boldsymbol{\Omega}} \times \overline{\mathbf{u}}_k + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \overline{\mathbf{u}}_k) \\ &= \ddot{\overline{\mathbf{u}}}_k\end{aligned}$$

From the previous equation, $\ddot{\mathbf{V}}_A$ acceleration vector is:

$$\ddot{\mathbf{V}}_A = \ddot{\mathbf{V}}_{AB} + \ddot{\mathbf{V}}_B + 2\sum_k \left(\dot{V}_{B,k}\dot{\mathbf{u}}_k\right) + \sum_k \left(V_{B,k}\ddot{\mathbf{u}}_k\right)$$

Solve for the acceleration in the B frame:

$$\ddot{\mathbf{V}}_B = \ddot{\mathbf{V}}_A - \ddot{\mathbf{V}}_{AB} - 2\sum_k \left(\dot{V}_{B,k}\dot{\mathbf{u}}_k\right) - \sum_k \left(V_{B,k}\ddot{\mathbf{u}}_k\right)$$

For use later, the equation is required to be in terms of forces ($F = ma$):

$$\mathbf{F}_B = \mathbf{F}_A - \mathbf{F}_{AB} - 2m\sum_k \left(\dot{V}_{B,k}\dot{\mathbf{u}}_k\right) - m\sum_k \left(V_{B,k}\ddot{\mathbf{u}}_k\right)$$

Before continuing, an identity needs to be proved, $\sum_k \left[A_k \left(\mathbf{\Omega} \times \mathbf{u}_k\right)\right] = \mathbf{\Omega} \times \mathbf{A}$, where $\mathbf{A}$ is some arbitrary vector.

$\mathbf{\Omega} \times \mathbf{A}$ reduces to:

$$\mathbf{\Omega} \times \mathbf{A} = \begin{vmatrix} x & y & z \\ 0 & 0 & \dot{\theta} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} -\dot{\theta} A_y \\ \dot{\theta} A_x \\ 0 \end{vmatrix}$$

$\sum_k \left[A_k \left(\mathbf{\Omega} \times \mathbf{u}_k\right)\right]$ reduces to:

$$\begin{aligned}
\sum_k \left[A_k \left(\mathbf{\Omega} \times \mathbf{u}_k\right)\right] &= A_x \left(\mathbf{\Omega} \times \mathbf{u}_x\right) + A_y \left(\mathbf{\Omega} \times \mathbf{u}_y\right) + A_z \left(\mathbf{\Omega} \times \mathbf{u}_z\right) \\
&= A_x \begin{vmatrix} x & y & z \\ 0 & 0 & \dot{\theta} \\ 1 & 0 & 0 \end{vmatrix} + A_y \begin{vmatrix} x & y & z \\ 0 & 0 & \dot{\theta} \\ 0 & 1 & 0 \end{vmatrix} + A_z \begin{vmatrix} x & y & z \\ 0 & 0 & \dot{\theta} \\ 0 & 0 & 1 \end{vmatrix} \\
&= A_x \begin{vmatrix} 0 \\ \dot{\theta} \\ 0 \end{vmatrix} + A_y \begin{vmatrix} -\dot{\theta} \\ 0 \\ 0 \end{vmatrix} + A_z \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 \\ \dot{\theta} A_x \\ 0 \end{vmatrix} + \begin{vmatrix} -\dot{\theta} A_y \\ 0 \\ 0 \end{vmatrix} \\
&= \begin{vmatrix} -\dot{\theta} A_y \\ \dot{\theta} A_x \\ 0 \end{vmatrix}
\end{aligned}$$

By inspection it can be seen that the two equations are equal. Taking the force equation and rotating frame B about the z-axis:

$$\begin{aligned}
\mathbf{F}_B &= \mathbf{F}_A - \mathbf{F}_{AB} - 2m \sum_k \left(\dot{V}_{B,k} \dot{\overline{\mathbf{u}}}_k\right) - m \sum_k \left(V_{B,k} \ddot{\overline{\mathbf{u}}}_k\right) \\
&= \mathbf{F}_A - \mathbf{F}_{AB} - 2m \sum_k \left[\dot{V}_{B,k} \left(\mathbf{\Omega} \times \overline{\mathbf{u}}_k\right)\right] - m \sum_k \left[V_{B,k} \left(\dot{\mathbf{\Omega}} \times \overline{\mathbf{u}}_k + \mathbf{\Omega} \times \left(\mathbf{\Omega} \times \overline{\mathbf{u}}_k\right)\right)\right]
\end{aligned}$$

Making some substitutions, $\mathbf{r}_B = \mathbf{V}_B$ and $\mathbf{v}_B = \dot{\mathbf{V}}_B$, gives the general form:

$$\mathbf{F}_B = \mathbf{F}_A - \mathbf{F}_{AB} - 2m\left(\mathbf{\Omega} \times \mathbf{v}_B\right) - m\left(\dot{\mathbf{\Omega}} \times \mathbf{r}_B\right) - m\left(\mathbf{\Omega} \times \left(\mathbf{\Omega} \times \mathbf{r}_B\right)\right) \tag{4.22}$$

$\mathbf{F}_B$	Force observed in frame B
$\mathbf{F}_A$	Force exerted on the mass in the inertial frame A
$\mathbf{F}_{AB}$	Force observed in frame B due to the translation of frame B in the inertial frame A
m:	Mass of the body
$\mathbf{r}_B$	Displacement of the mass in frame B
$\mathbf{V}_B$	Velocity of the mass in frame B
$\mathbf{\Omega}$	Rotation rate about the z-axis, $\left[0, 0, \dot{\theta}\right]^T$
$2m\left(\mathbf{\Omega} \times \mathbf{v}_B\right)$	Coriolis fictitious force observed in frame B
$m\left(\dot{\mathbf{\Omega}} \times \mathbf{r}_B\right)$	Euler fictitious force observed in frame B
$m\left(\mathbf{\Omega} \times \left(\mathbf{\Omega} \times \mathbf{r}_B\right)\right)$	Centrifugal fictitious force observed in frame B

For the rotating frame:

$$\mathbf{\Omega} \times \mathbf{v}_B = \begin{vmatrix} -v_{B,y}\dot{\theta} \\ v_{B,x}\dot{\theta} \\ 0 \end{vmatrix}$$

$$\dot{\mathbf{\Omega}} \times \mathbf{r}_B = \begin{vmatrix} -r_{B,y}\ddot{\theta} \\ r_{B,x}\ddot{\theta} \\ 0 \end{vmatrix}$$

$$\mathbf{\Omega} \times \left(\mathbf{\Omega} \times \mathbf{r}_B\right) = \mathbf{\Omega} \times \begin{vmatrix} -r_{B,y}\dot{\theta} \\ r_{B,x}\dot{\theta} \\ 0 \end{vmatrix} = \begin{vmatrix} -r_{B,x}\dot{\theta}^2 \\ -r_{B,y}\dot{\theta}^2 \\ 0 \end{vmatrix}$$

Substituting into equation 4.22:

$$\begin{aligned} F_{B,x} &= F_{A,x} - F_{AB,x} + 2\,m\,v_{B,y}\dot{\theta} + m\,r_{B,y}\ddot{\theta} + m\,r_{B,x}\dot{\theta}^2 \\ F_{B,y} &= F_{A,y} - F_{AB,y} - 2\,m\,v_{B,x}\dot{\theta} - m\,r_{B,x}\ddot{\theta} + m\,r_{B,y}\dot{\theta}^2 \\ F_{B,z} &= F_{A,z} - F_{AB,z} \end{aligned}$$

If $\theta(t)$ is the constant rotation, this can be rewritten a function of time with a phase offset ($\phi(t)$):

$$\begin{aligned}
\theta(t) &= \Omega_z\, t + \phi(t) \\
\dot{\theta}(t) &= \Omega_z + \dot{\phi}(t) \\
\ddot{\theta}(t) &= \ddot{\phi}(t)
\end{aligned}$$

And $\left[\dot{\theta}(t)\right]^2$ is:

$$\begin{aligned}
\left[\dot{\theta}(t)\right]^2 &= \left(\Omega_z + \dot{\phi}(t)\right)\left(\Omega_z + \dot{\phi}(t)\right) \\
&= {\Omega_z}^2 + 2\Omega_z\dot{\phi}(t) + \left(\dot{\phi}(t)\right)^2
\end{aligned}$$

Then substituting into the individual force equations:

$$\begin{aligned}
F_{B,x} &= F_{A,x} - F_{AB,x} + 2\,m\,v_{B,y}\Omega_z + 2\,m\,v_{B,y}\dot{\theta}(t) + m\,r_{B,y}\ddot{\phi}(t) + m\,r_{B,x}{\Omega_z}^2 + 2\,m\,r_{B,x}\Omega_z\dot{\theta}(t) + m\,r_{B,x}\left(\dot{\phi}(t)\right)^2 \\
F_{B,y} &= F_{A,y} - F_{AB,y} - 2\,m\,v_{B,x}\Omega_z + 2\,m\,v_{B,x}\dot{\theta}(t) - m\,r_{B,x}\ddot{\phi}(t) + m\,r_{B,y}{\Omega_z}^2 + 2\,m\,r_{B,y}\Omega_z\dot{\theta}(t) + m\,r_{B,y}\left(\dot{\phi}(t)\right)^2 \\
F_{B,z} &= F_{A,z} - F_{AB,z}
\end{aligned}$$

If the phase angle (θ) is constant, then the derivative are zero, $\dot{\theta} = \ddot{\theta} = 0$. The equations simplify to:

$$\begin{aligned}
F_{B,x} &= F_{A,x} - F_{AB,x} + 2\,m\,v_{B,y}\Omega_z + m\,r_{B,x}{\Omega_z}^2 \\
F_{B,y} &= F_{A,y} - F_{AB,y} - 2\,m\,v_{B,x}\Omega_z + m\,r_{B,y}{\Omega_z}^2 \\
F_{B,z} &= F_{A,z} - F_{AB,z}
\end{aligned} \tag{4.23}$$

4.3 Extracting the Rate Measurement

There are two channels of data from the gyro: drive and sense. The drive signal is read in on channel 0 (A0). The sense signal is read in on channel 1 (A1). Both channels are run through a bandpass filter set to 2 to 5 Hz. These signals are then mixed and low-pass filters to get the in-phase response. The in-phase signal (I) is the gyroscope output. Figure 4.8 shows this block diagram.

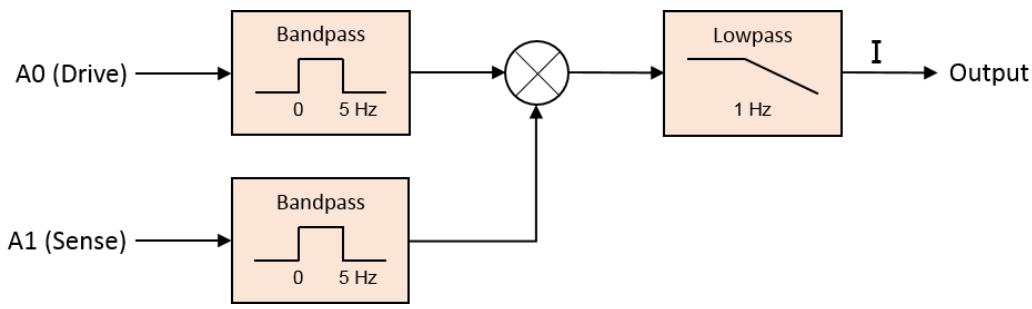

Figure 4.8: Rate Measurement Block Diagram

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5

Mechanical System

The entire gyroscope was constructed out of Lego building blocks except for a few exceptions. A wooden base was added to provide mounting wholes to attached the gyroscope to the precision rate table. The weight for M_1 was achieved with stainless steel bolts. Lego wheels were used to lower the friction of the moving parts. There are two moving parts: M_1 and M_2 as show in figure 5.1. M_1 can only move along the drive axis and M_2 can only move along the sense axis. Table 5.1 lists the masses. Figure 5.6 is a photo of one of the entire system. Figure 5.2 depicts the drive axis (x-axis). Figure 5.3 is a photo of M_1. The bolts were secured using a hot glue gun.

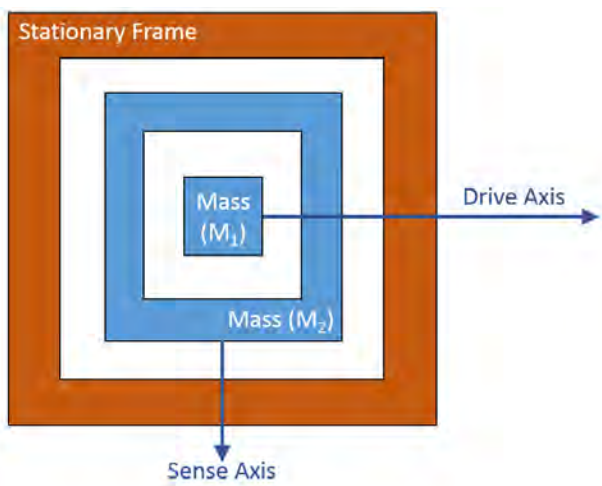


Figure 5.1: Mechanical Overview

Parameter	Units	Measurement
M_1	kg	0.296
M_2	kg	0.232
$M_1 + M_2$	kg	0.528

Table 5.1: Mechanical Masses

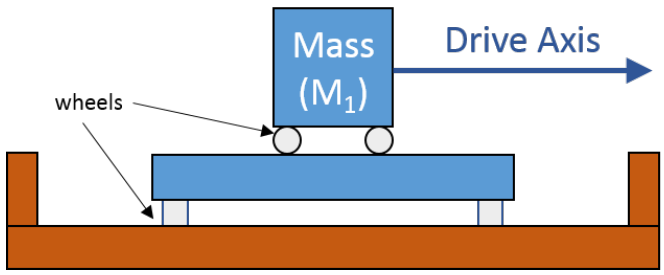


Figure 5.2: Mechanical Drive Axis

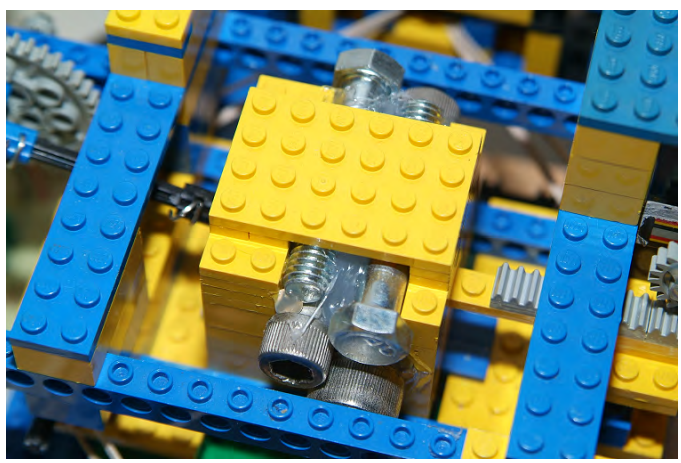


Figure 5.3: M_1 Photo

The sense axis (y-axis) was created using rubber bands and placing the sense assembly on wheels. The rubber bands allowed the sense axis to have a resonant frequency at the frequency of the drive axis. Figure 5.4 depicts the sense axis. Figure 5.5 is a photo of one of the rubber bands. Figure 5.6 is a photograph of the entire Lego gyroscope with the axis labeled.

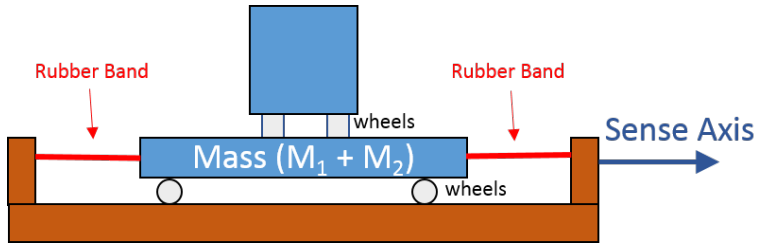


Figure 5.4: Mechanical Sense Axis

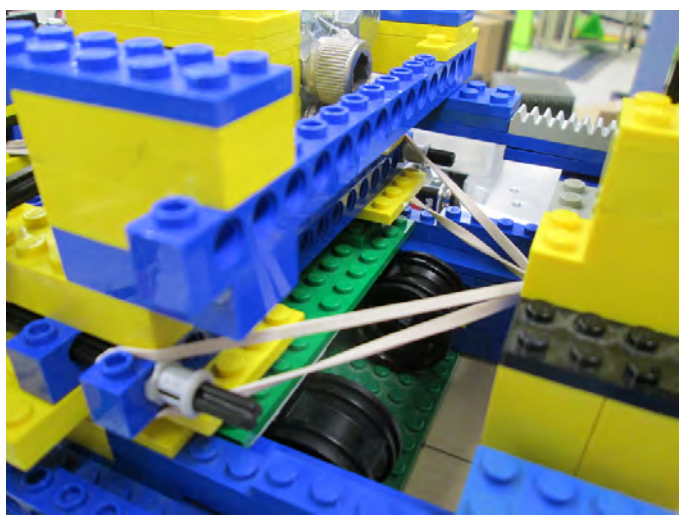


Figure 5.5: Rubber Band Photo

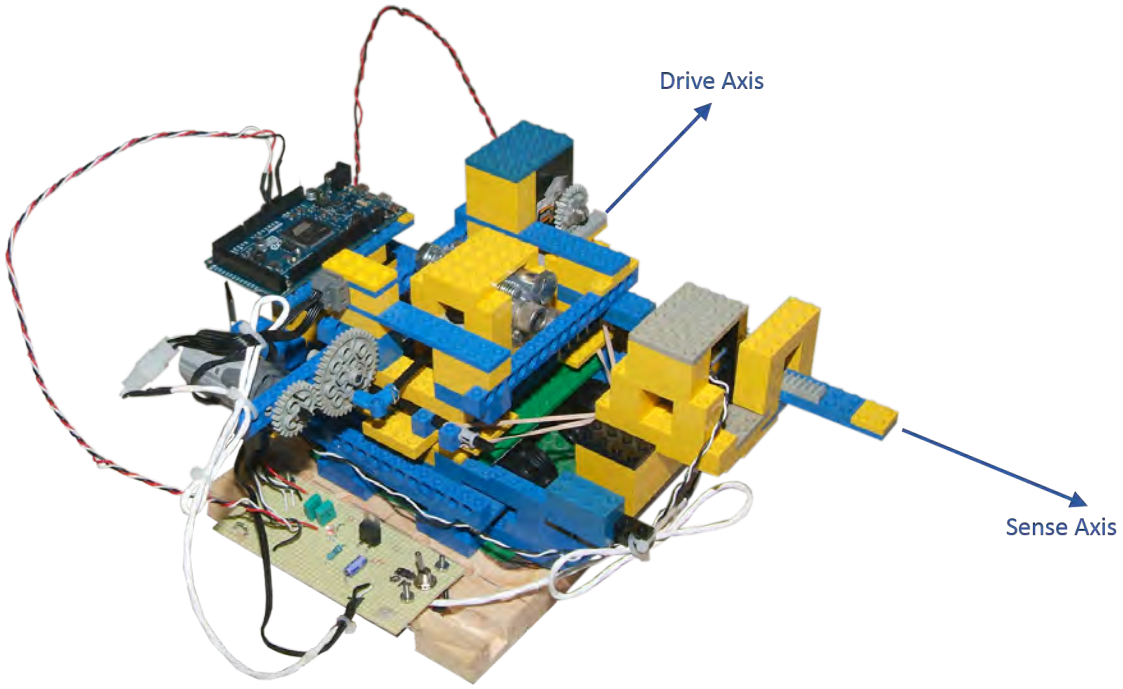


Figure 5.6: Mechanical Photo

6

Electromechanical System

For the Lego gyroscope, a direct linkage from the motor to the mass was used to create the oscillation along the drive axis. This design is shown in figure 6.1. A Lego motor was connected directly to a Lego gear. One side of a Lego shaft was connected to the edge of the gear and the other side to the mass. In this configuration, the constantly rotating motor was used to make the mass oscillate back and forth. The frequency of the oscillition was chosen to be at the resonant frequency of the sense axis. The Lego pieces connecting the shaft to the gear and to the mass did require a special non-Lego pin and glue to keep from separating over time. Figure 6.2 is a photograph of the drive mechanism.

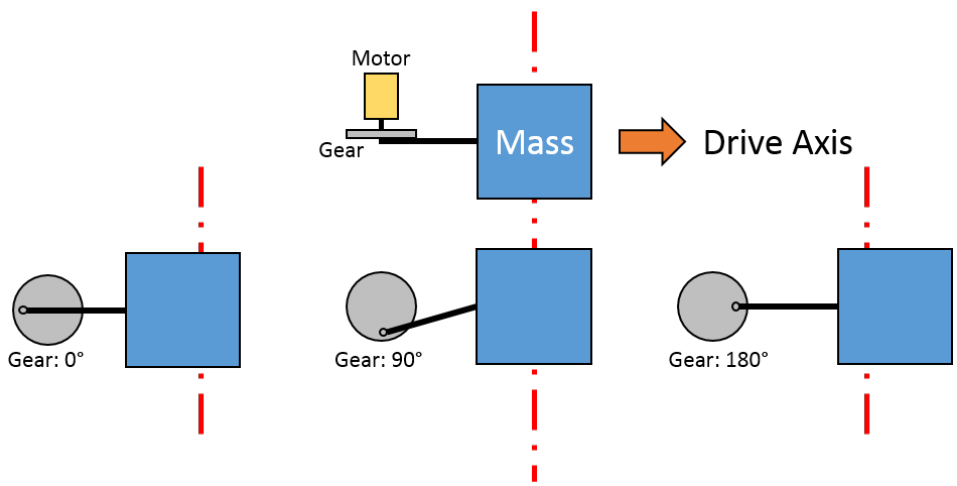
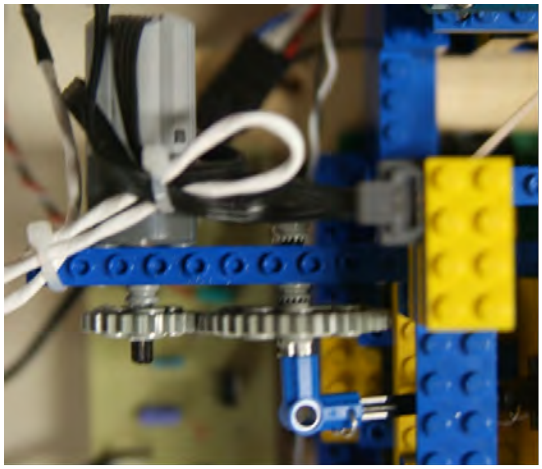
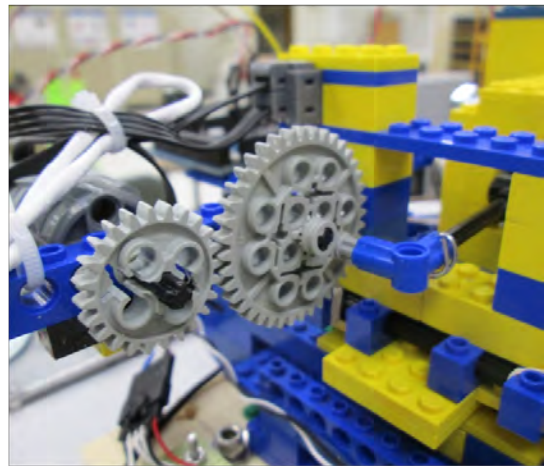


Figure 6.1: Drive Axis Setup



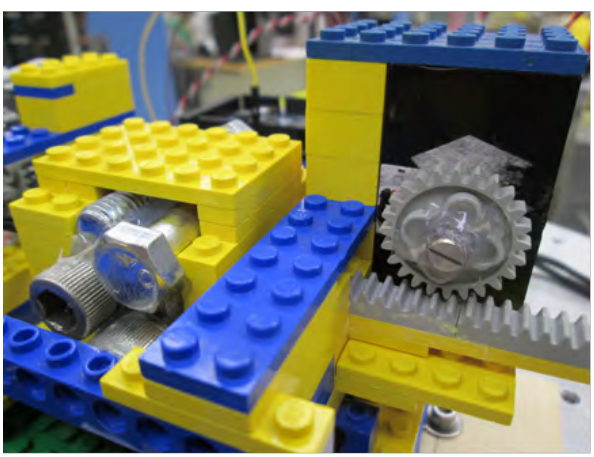
Top View



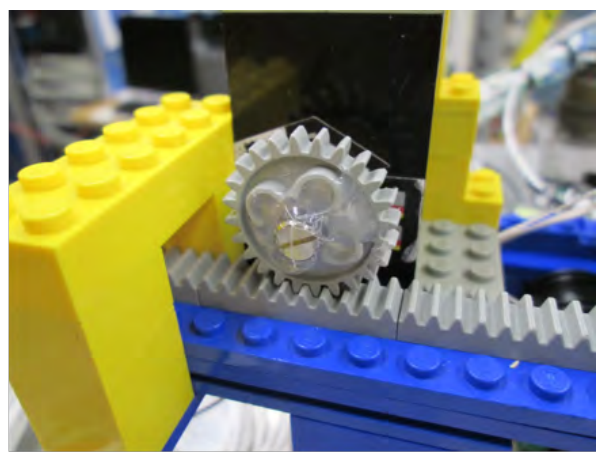
Side View

Figure 6.2: Drive Axis Configuration

A potentiometer was used to convert the mechanical motion to and electrical signal. Two $20K\Omega$ potentiometers were used on each axis. Figure 6.3 shows the configuration of each axis. The linear motion of the bottom piece rotates the gear. This rotation is measured as a change in the resistance. Figure 6.4 show a side view of the potentiometer.



Drive Axis



Sense Axis

Figure 6.3: Axis Pickups

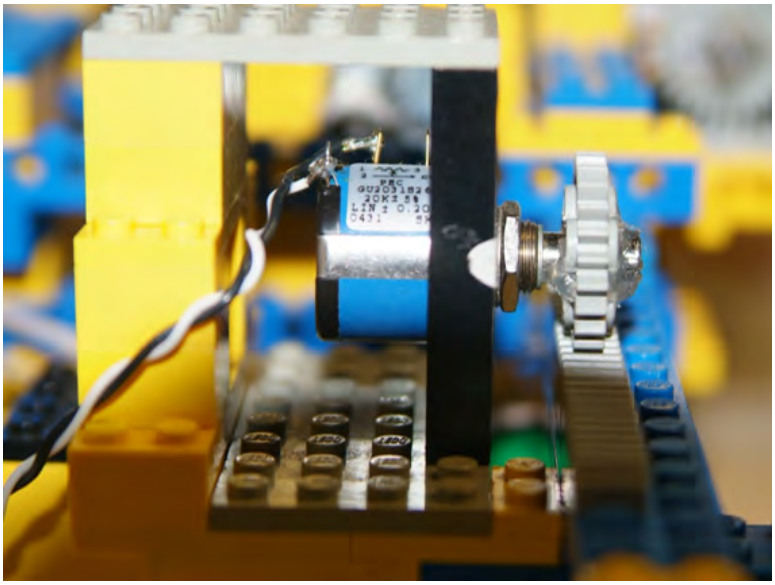


Figure 6.4: Side view of Potentiometer

A test was conducted to see what the change in resistance of potentiometer would be starting at different center ohms. The travel distance was ±20 millimeters for the drive direction (x-axis) and ±25 millimeters for the sense direction (y-axis). Table 6.1 contains this data. The conversion factor was then calculated from this data. For simplicity, a single conversion factor of 22 Ω per millimeter was selected.

$$\text{The conversion factor} = \frac{\Omega \text{ Range}}{\text{Distance Traveled}}$$

Center Ω	X Range Ω	Y Range Ω	X Ω/mm	Y Ω/mm
1000	566, 1395	388, 1593	21	24
2000	1670, 2480	1410, 2660	20	25
10000	9450, 10250	9430, 10650	20	24
15000	14540, 15410	14420, 15640	22	24

Table 6.1: Potentiometer Values Across the Travel Range

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7

Electrical System

The electronics for the gyroscope was a simple design as seen in figure 7.1. The motor moves the drive axis (x-axis). The distance traveled is converted to a voltage by way of a voltage divider. If the gyro is in a rotating frame about the z-axis, the Coriolis force will move the sense axis (y-axis). This motion is also converted to a voltage. Both voltages will then be sampled and read into the microcontroller by way of two 12-bit analog-to-digital (A/D) converter. The microcontroller used was an Arduino Due with channels A0 and A1 used for the conversion. The data was then sent to a computer for processing over a RS232 serial link. Figure 7.2 shows the electric schematic for the system. Figure 7.3 is the photograph of the circuit.

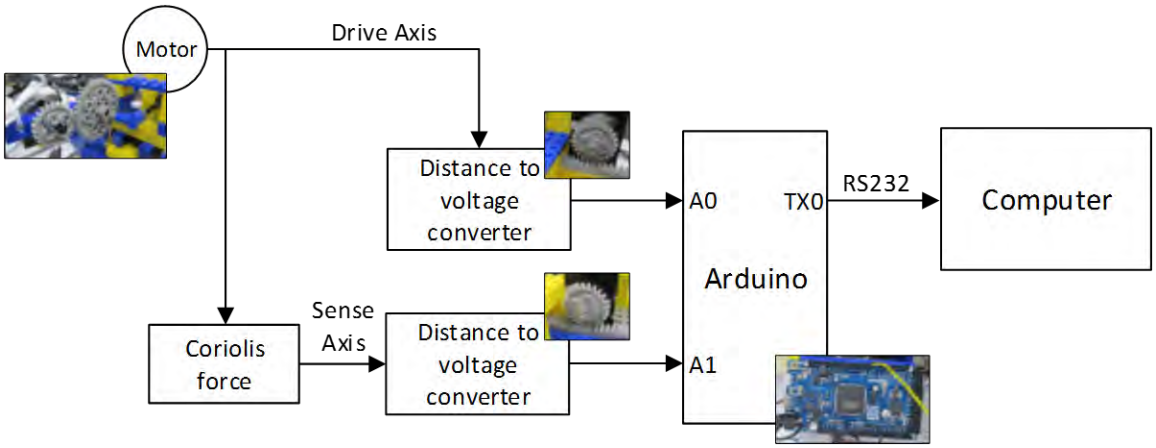

Figure 7.1: Electrical Block Diagram

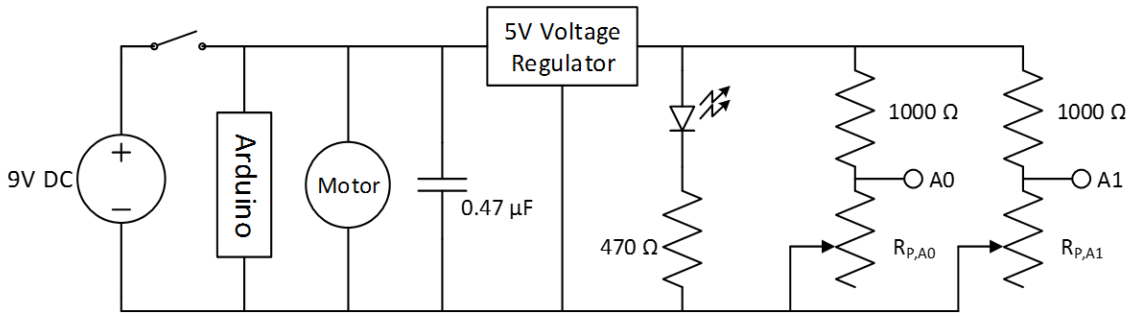

Figure 7.2: Electrical Schematic

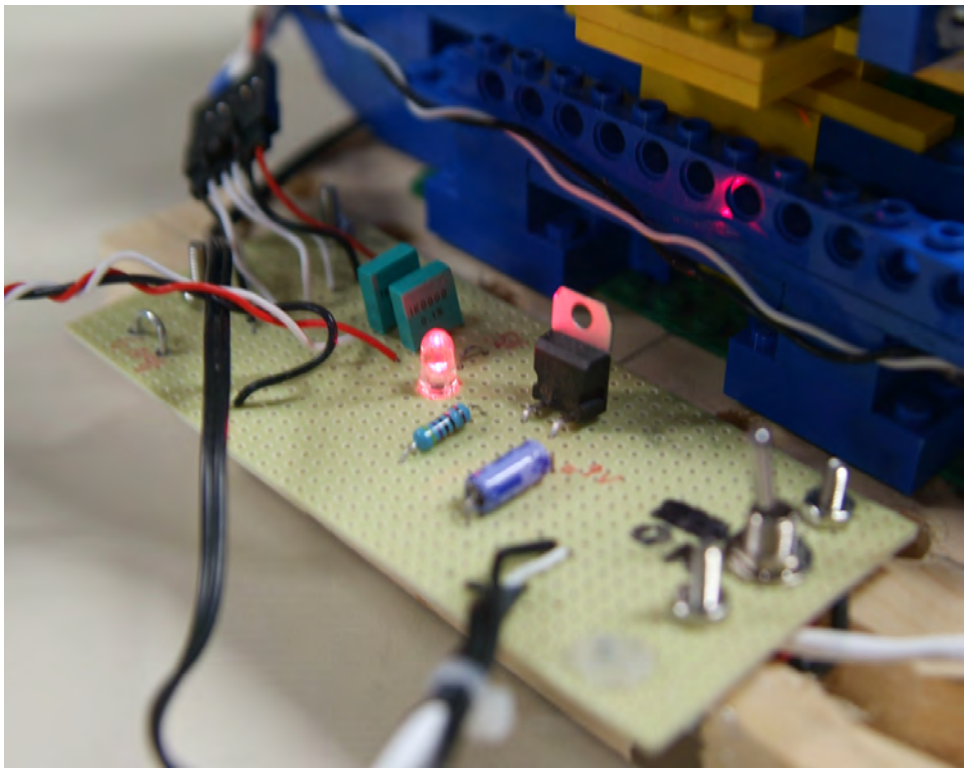

Figure 7.3: Electrical Circuit Photo

Figure 7.4 shows the circuit to convert ohms to volts so that it can be read into the Arduino's A/D. The A/D's range is 0 to 3.3 volts.

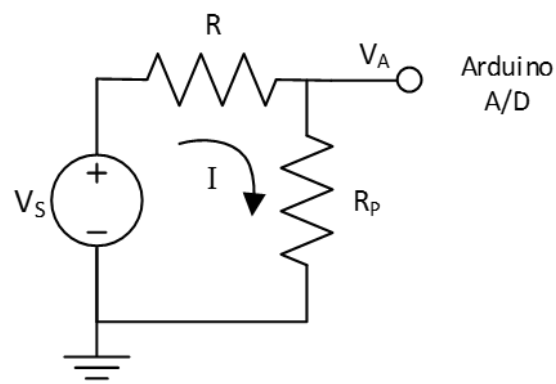

Figure 7.4: Ohms to Volts

The following equations describe the circuit.

$$
\begin{aligned}
V_S &= I(R + R_P) \text{ and } V_A = I\,R_P \\
V_S &= \frac{V_A}{R_P}(R + R_P) \\
V_A &= \frac{V_S\,R_P}{(R + R_P)}
\end{aligned}
$$

Some of the parameters are know:

$$
\begin{aligned}
R &= 1000\Omega(\text{selected based on inventory}) \\
R_P &= R_B + \Delta R \\
R_B &\text{ is the potentiometer bias} \\
\Delta R &\text{ is the potentiometer range of motion about the bias} \\
\Delta R &= \pm 550\Omega \left(\frac{22\Omega}{\text{mm}} \;\; @ \;\; \pm\, 25\text{mm} \right)
\end{aligned}
$$

In order the maximize the resolution of the 12-bit A/D, the following conditions need to be met:

$$(1)\ \Delta R = -575\Omega \text{ when } V_A = 0V$$
$$(2)\ \Delta R = +575\Omega \text{ when } V_A = 3.3V$$

Using the equations from before and rearranging and replacing R_P:

$$V_A = \frac{V_S\, R_P}{(R + R_P)}$$
$$V_S\, R_P = V_A(R + R_P)$$
$$V_S(R_B + \Delta R) = V_A\, R + V_A(R_B + \Delta R)$$

Using the second condition:

$$V_S(575 + 575) = 1000(3.3) + (3.3)(575 + 575)$$
$$1150V_S = 7095$$
$$V_S = 6.17$$

For simplicity, V_S will be 5 volts. If $V_S = 5V$, then the following will be true:

$$R_B = 971\,\Omega$$
$$\Delta R = \pm 971\Omega \text{ range}$$

Then to convert the volotage, V_A, to a distance, the following equation can be used:

$$\text{meters} = \frac{1971V_A - 4855}{110000 - 22000V_A}$$

8

Software

Software was written to simulate the gyroscope and also to analysis the real and simulated data. The "Arduino ComPortScope" software is shown in figure 8.1. This software reads in the data from the Arduino microcontroller into a data file on the computer. The "Lego Gyro Sim" software is shown in figure 8.2. This software not only simulated the gyroscope, it also processed the raw data into sensed rate. Figure 8.3 is an example of processed simulated data from a simulated multi-rate test.

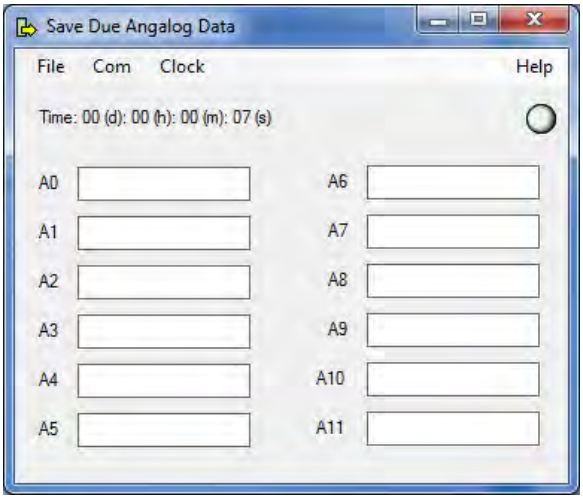

Figure 8.1: Arduino ComPortScope Software Screenshot

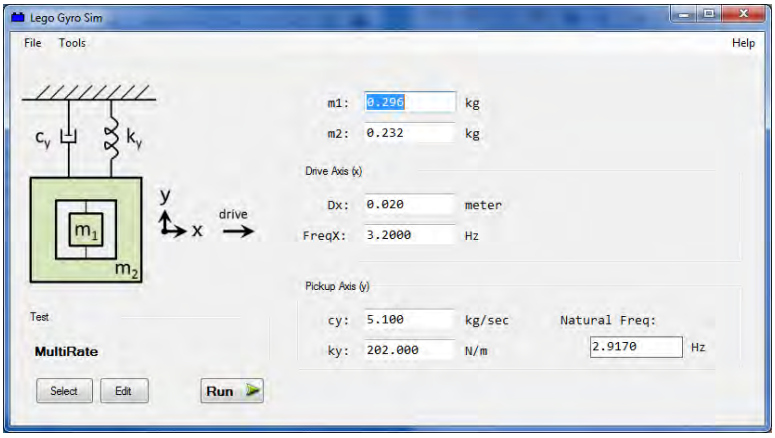

Figure 8.2: Lego Gyro Sim Software Screenshot

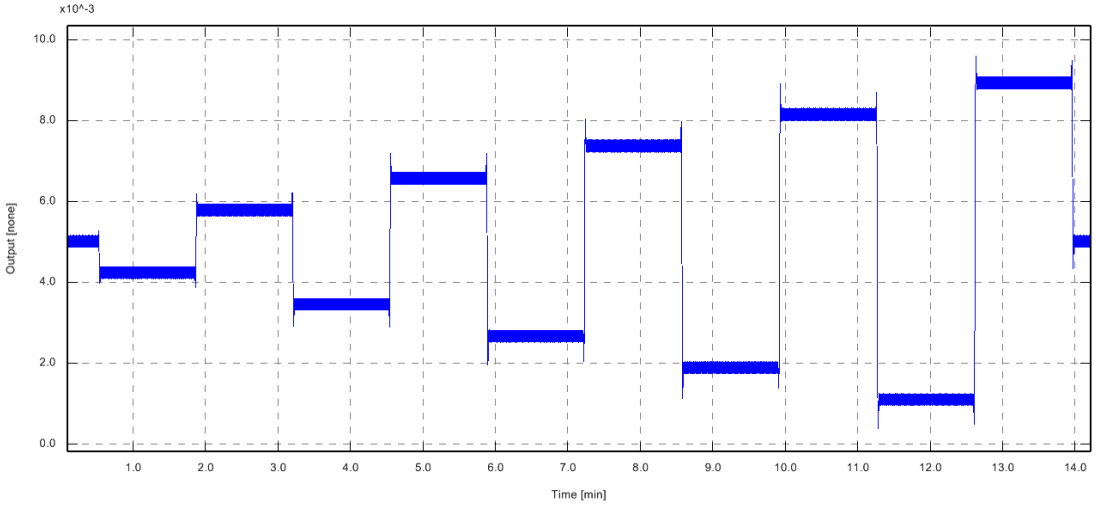

Figure 8.3: Example of Processed Simulated Multi-Rate Test

9

Testing Effort

Table 9.1 contains the list of tests conducted for this project. Figure 9.1 is a photograph of the Lego gyro on a cordless drill at the house. The cordless drill was used as a rate table to verify the gyro actually did measure rate. Figure 9.2 shows the data for this test. Figure 9.3 is a photograph of the data on the precision rate table at NRDC. Figure 9.4 is the data from test 1003.

Number	Description
1001	Test conducted at house with a drill
1002	Multi-rate test at NRDC, problem with the rate table
1003	Multi-rate test at NRDC
1004	Stability test at NRDC
1005	Impulse response test at house

Table 9.1: Tests Conducted

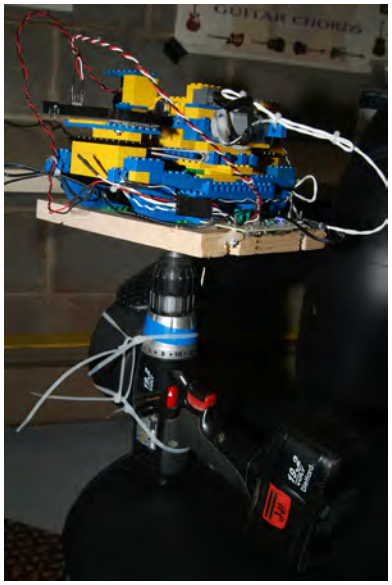

Figure 9.1: At the House on the Drill

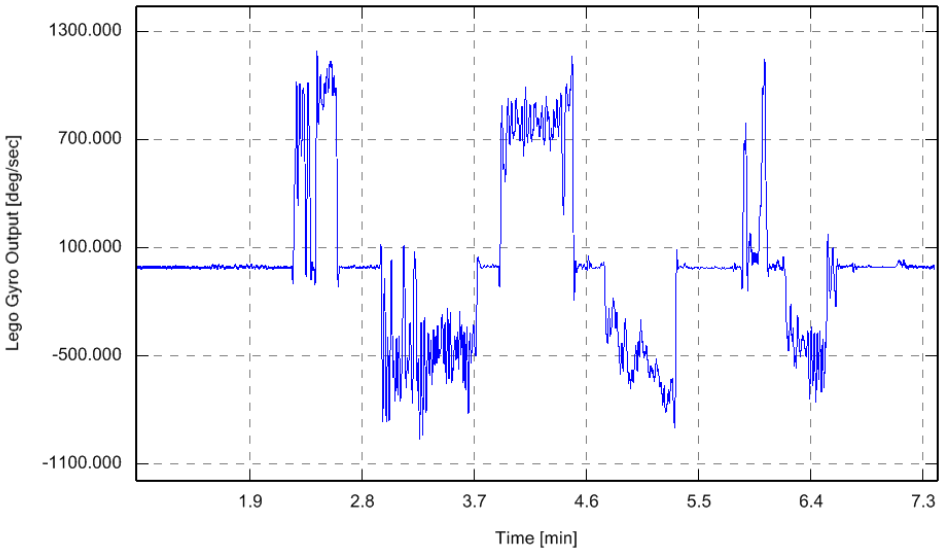

Figure 9.2: Test 1001 Time Data

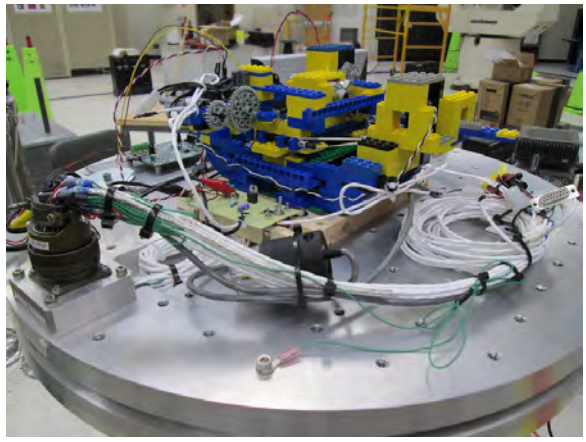

Figure 9.3: On the Rate Table

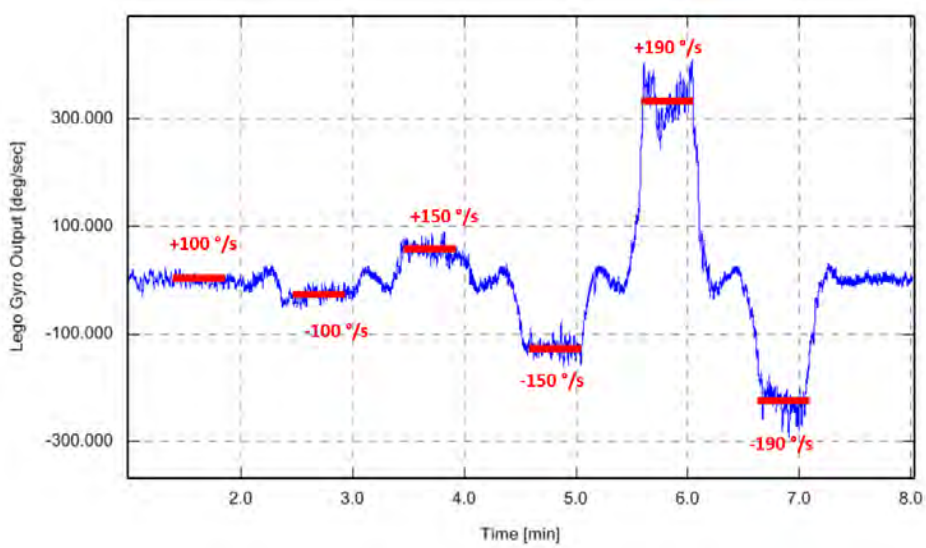

Figure 9.4: Test 1003 Time Data

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10

Results

10.1 Homogeneous Solution Results

The homogeneous solution to the differential equation along the sense axis is useful in determining the system's parameters. The mass (m) of the system for the Lego gyroscope was determined by measurement with a scale. The spring (k) and damper (c) terms were determined by physically moving the Lego gyroscope's mass to the left or right along the sense axis then letting go; the system would then oscillate with an exponential decay. This type of response is expected and shown by the solution to the homogeneous equation. It is also noted that the Lego gyroscope was created purposely this way. The homogeneous equation is then set to the initial conditions and the constants (C_1 and C_2) are solved for.

$$y_h = C_1 e^{\frac{-t}{\tau}} cos(\omega t) + C_2 e^{\frac{-t}{\tau}} sin(\omega t)$$

$$\dot{y}_h = \frac{-t}{\tau} C_1 e^{\frac{-t}{\tau}} cos(\omega t) - C_1 \omega e^{\frac{-t}{\tau}} sin(\omega t) - \frac{t}{\tau} C_2 e^{\frac{-t}{\tau}} sin(\omega t) + C_2 \frac{-t}{\tau} e^{\frac{-t}{\tau}} cos(\omega t)$$

$$\tau = \frac{2m}{c}$$

$$\omega = \frac{\sqrt{4mk - c^2}}{2m}$$

The initial conditions are defined as:

$$y(0) = A \text{ [meters]}$$
$$\dot{y}(0) = B \text{ [meters/sec]}$$

where A and B are given. For the test $y(0) = 0.035$ and $\dot{y}(0) = 0$. The solutions to C_1 and C_2 are:

$$C_1 = A \tag{10.1}$$
$$C_2 = \frac{B + \frac{A}{\tau}}{\omega} \tag{10.2}$$

Actual measured response of the system when "let go" was collected. This data was then fit to the analytical solution derived in equations 10.1 to 10.2 to determine the spring and damper terms of the Lego gyroscope. Two tests were executed. Test-A started the mass over at -44.1 mm. Test-B started the mass over at +35.0 mm. These parameters are summarized in table 10.1.

Parameter	Units	Test-A	Test-B
Start Time	sec	24.60	28.00
Initial Y	mm	-44.10	+35.00
"Let Go" Time	sec	24.96	28.12
End Time	sec	25.80	28.90

Table 10.1: Homogeneous Solution Fit Parameters

A range of spring (k) and damper (c) terms were used to create a fit to the measured data. At each step, the standard deviation of residuals were calculated from the measured data to the fit data. This created a two-dimensional plot of k, c, and standard deviation of residuals. Figures 10.1 and 10.2 show these two-dimensional plots. The results from the plots are summarized in table 10.2. Test-A and Test-B showed matching results.

Parameter	Units	Test-A	Test-B	Average
c	kg/sec	4.9	5.3	5.1
k	N/meter	192	212	202

Table 10.2: Homogeneous Solution Fit Solution

Figure 10.3, and 10.4 show the measured data for each test and and best fit as determined from figures 10.1 and 10.2.

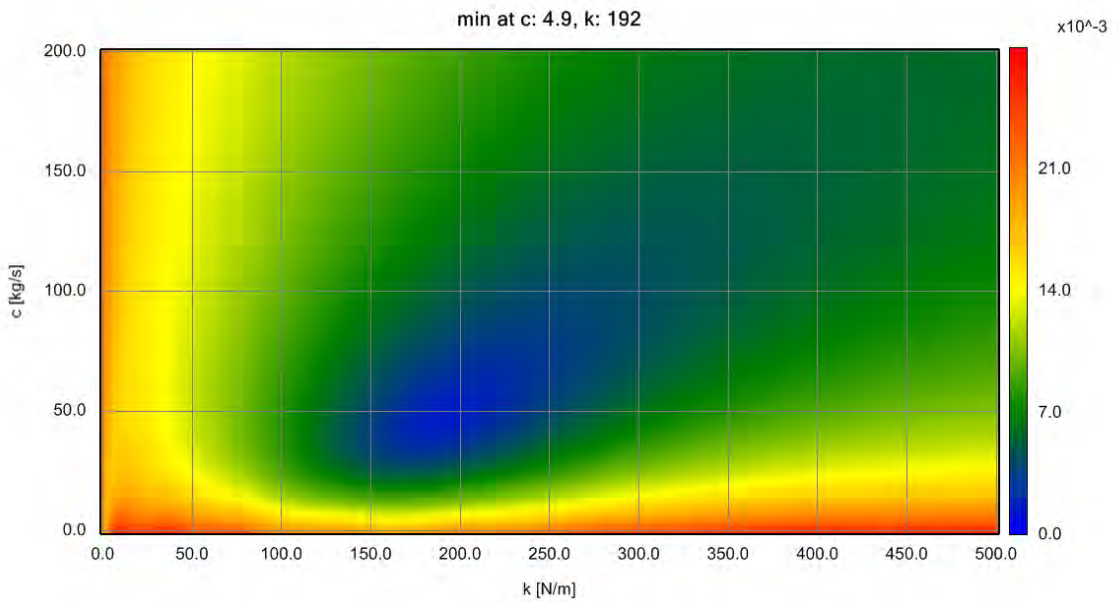

Figure 10.1: Two dimensional plot of residuals from the homogeneous solution fit for Test-A

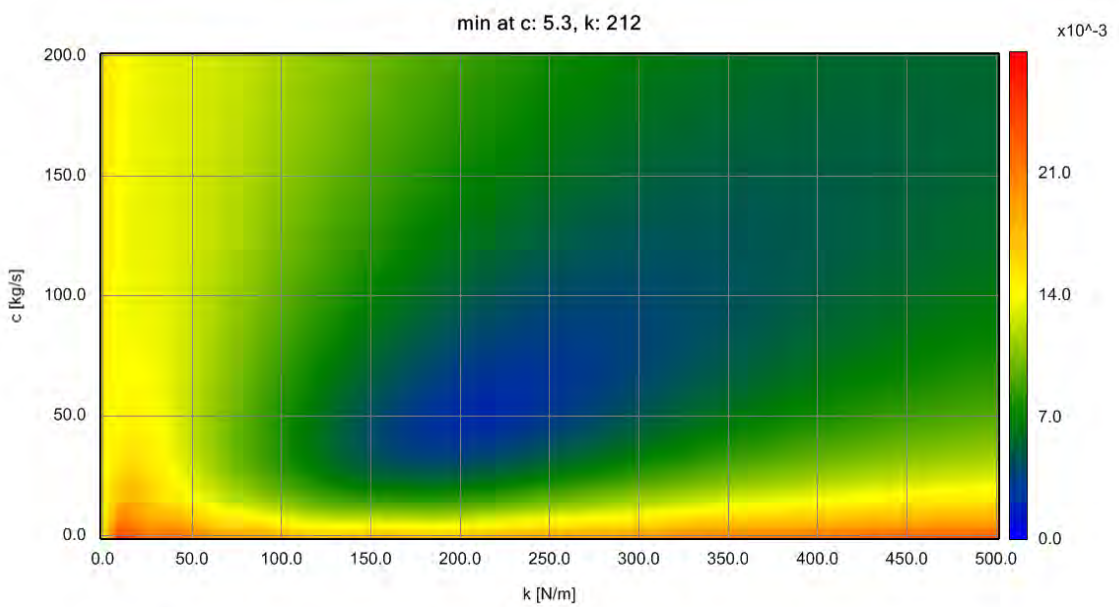

Figure 10.2: Two dimensional plot of residuals from the homogeneous solution fit for Test-B

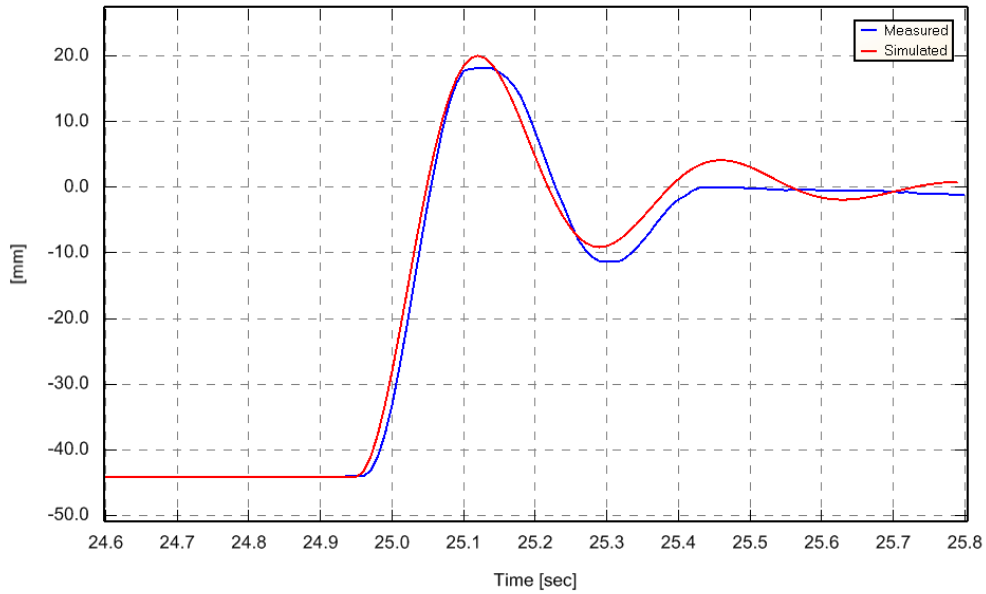

Figure 10.3: Measured data from the Lego gyroscope (blue) and best homogeneous solution fit (red) for Test-A

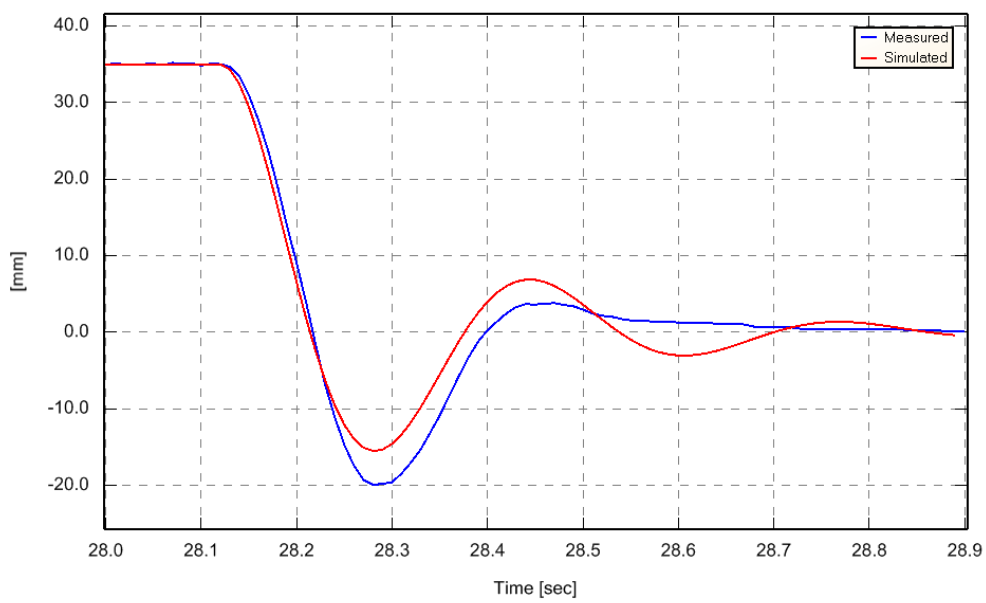

Figure 10.4: Measured data from the Lego gyroscope (blue) and best homogeneous solution fit (red) for Test-B

10.2 Bias-instability and ARW Results

A test was conducted to measure the bias-instability and the Angle Random Walk (ARW). The sensor was stationary and data was saved at 100 Hz for 1.4 hours. Figure 10.5 shows the data over time. Figure 10.6 shows the Allan-deviation of this data in deg/sec. The minimum of this plot is the bias-instability after it divided by 0.6 and then multiplies by 3600 to get it into deg/hr. Figure 10.7 shows Power Spectral Density (PSD) plot of the data. From this plot the gyroscope ARW can be calculated. The mean of the selected area is then multiplies by 60 to get it into deg/$\sqrt{\text{hr}}$. Table 10.3 list these results.

Parameter	Units	Result
Bias	deg/hr	14509
Standard Deviation	deg/hr 1-σ	42414
Bias-instability	deg/hr	11370
ARW	deg/$\sqrt{\text{hr}}$	1063

Table 10.3: Bias-instability and ARW Results

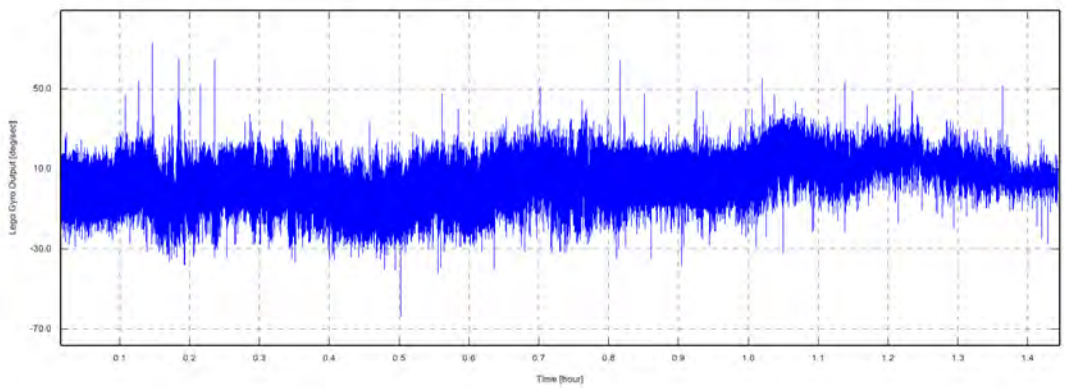

Figure 10.5: Data verses Time

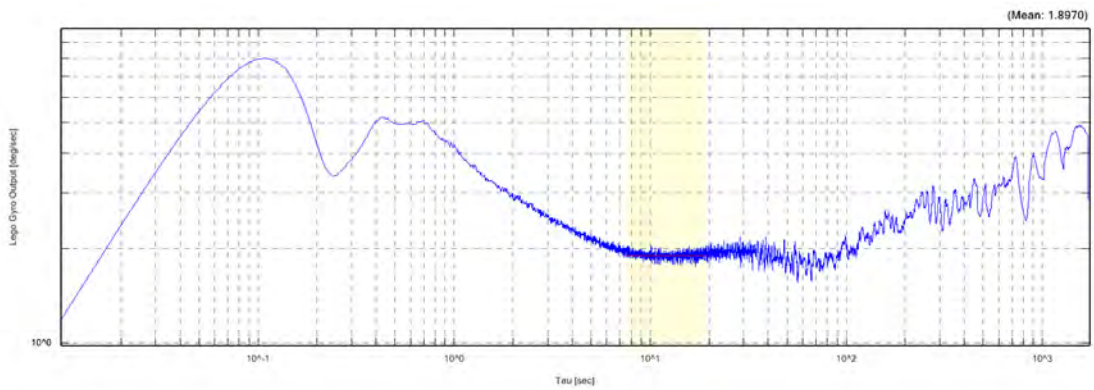

Figure 10.6: Allan-Deviation

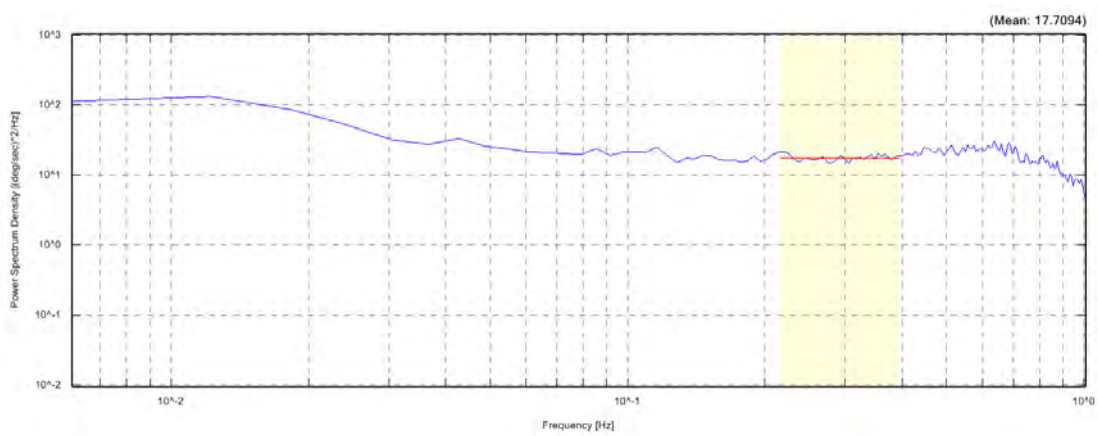

Figure 10.7: PSD Plot

10.3 Scale Factor Results

A test was conducted with precision rate table. The table was rotated at ±100 deg/sec, ±150 deg/sec, and ±190 deg/sec. Data was recorded throughout the test effort. From the data collected, a linear fit was applied to the gyro output versus the input rate. This fit is show in Figure 10.8. The Lego gyroscope scale factor was determined to be 33333 deg/sec/output. The Lego gyroscope in-run bias was determined to be -143 deg/sec. The fit also shows a large scale factor non-linearity which was expected because of the Lego design using rubber bands for the springs.

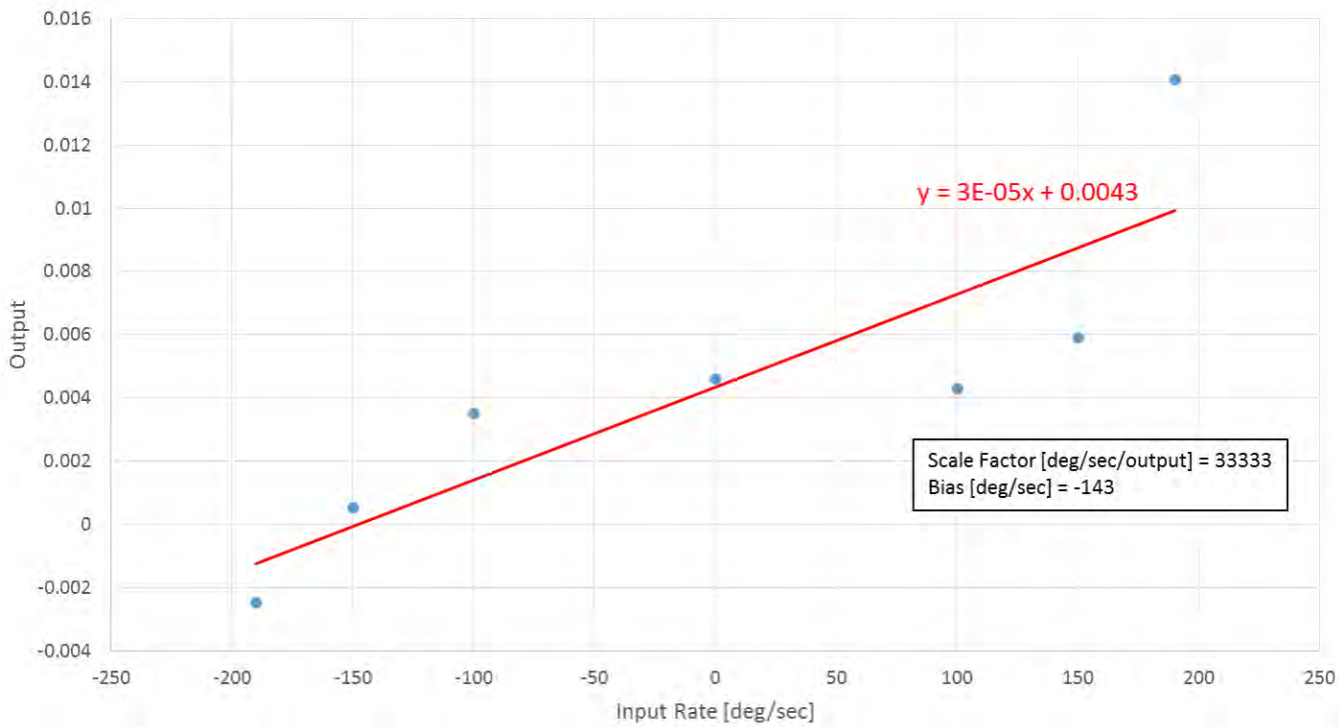

Figure 10.8: Lego Gyroscope Output vs Input Rate

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