DRAFT WORKING PAPER, OCTOBER 2006

Decomposing Congestion and Reliability

Seth Blumsack, Marija Ilić, and Lester B. Lave

***Abstract*—Policy surrounding the North American transmission grid, particularly in the wake of electric-industry restructuring and following the blackout of August, 2003, has treated network congestion and network reliability as if they were separable and independent system attributes. Except for a few special cases, congestion and reliability are not independent, and may not even be separable in any meaningful way. Using the DC power flow model with linear ATC, we provide a method for decomposing a change in network topology into a congestion effect and a reliability effect. We provide analytical expressions describing the topological conditions under which a given network addition or outage will affect congestion and reliability, and prove some sufficiency conditions and some necessary conditions for congestion and reliability to be independent. These include (i) the network is series-parallel; (ii) demand is completely price-inelastic; (iii) all customers value reliability identically; and (iv) the grid operator does not discriminate among customers when forced to physically ration consumption.**

***Index Terms*—Braess Paradox, Wheatstone network, merchant transmission, available transfer capability, reliability, congestion, transmission investment.**

I. NOMENCLATURE

NL = Number of lines in the network
NB = Number of buses in the network
S_{ij} = Transmission line connecting buses i and j
B_{ij} = Susceptance of the link connecting buses i and j
X_{ij} = Reactance of the link connecting buses i and j
θ_i = Phase angle at the ith bus
P_i = Net real power injection at the ith bus; positive for net generation and negative for net withdrawal
P_{Li} = Real power demand at the ith bus
P_{Gi} = Real power demand at the ith bus
ρ_{li} = Power transfer distribution factor along line l with respect to a network resource at bus i.
δ_{ij} = Phase angle difference between buses i and j
F_{ij} = Real power flow between buses i and j
Ψ_i = Set of lines directly connected to bus i
π_i = Nodal price at bus i
μ_{ij} = Shadow price of transmission between buses i and j
C_i = Total cost function at the ith bus.
ATC_i = Available transfer capacity into bus i.
$\mathbf{B}$ = ($NB \times NB$) system susceptance matrix
$\mathbf{B}^{\mathbf{diag}}$ = ($NL \times NL$) diagonal matrix of line susceptances
$\mathbf{N}$ = ($NB \times NB$) node-node adjacency matrix
$\mathbf{A}$ = ($NB \times NL$) system node-line adjacency matrix
$\mathbf{P}$ = ($NB \times 1$) vector of bus injections
$\mathbf{F}$ = ($NL \times 1$) vector of line flows
θ = ($NB \times 1$) vector of bus angles
δ = ($NL \times 1$) vector of bus angle differences
ρ = ($NL \times NB$) matrix of power transfer distribution factors

II. INTRODUCTION

Restructuring in the U.S. electric power sector has encouraged investment by the non-utility, or "merchant" sector. Initially, investment activity in the merchant generation sector was high, with 55 GW of mostly gas-fired capacity added between 1995 and 2002 (Joskow 2005). Merchant transmission has been far less successful, existing for the most part only in theory. Language supporting merchant transmission exists both in regulatory documents (such as section 1221 of the 2005 Energy Policy Act) and in the tariffs of the PJM, New York ISO, and ISO New England regional transmission operators. Actual non-utility transmission investment has been minimal.

Merchant transmission and market-based transmission investment were originally synonymous due to the dis-integration of the traditional electric utility, and the takeover of the system planning function by ISOs and RTOs. Persistent differences in locational marginal prices (LMP) would serve as signals for investors. Compensation for system upgrades would come in the form of contracts entitling the investor to some share of the congestion rent, related to the incremental capacity created by the upgrade. Proposed transmission congestion contracts initially took the form of point-to-point financial transmission rights (FTR, Hogan 1992) or line-by-line "flowgate" rights (Chao and Peck 1996, Oren 1997). More recent market-based compensation mechanisms include the admittance-rights formulation of Gribik et. al., and the LMP/megawatt-mile formulation of Apt and Lave (2004).

The use of FTRs to encourage transmission investment has been supported by the analysis of Bushnell and Stoft (1996, 1997). They demonstrate that if incremental FTRs are allocated according to Hogan's "feasibility allocation rule,"

This work was funded by a grant to the Carnegie Mellon Electricity Industry Center from EPRI and the Alfred P. Sloan foundation, and by the Tennessee Valley Authority. Any opinions and errors are those of the authors and should not be ascribed to the grantors

Seth Blumsack is a postdoctoral researcher at the Tepper School of Business, Carnegie Mellon University, Pittsburgh PA 15213 (email: blumsack@cmu.edu).

Marija Ilić is Professor of Electrical and Computer Engineering and Enginering and Public Policy, Carnegie Mellon University, Pittsburgh PA 15213 (email: milic@ece.cmu.edu).

Lester B. Lave is the Harry and James Higgins Professor of Economics and Finance, Tepper School of Business, Carnegie Mellon University, Pittsburgh PA and Co-Director of the Carnegie Mellon Electricity Industry Center (email: lave@cmu.edu).

(Hogan 1992, 1993), and if other economic assumptions hold, then merchant transmission can be economically efficient – all socially beneficial network investments will also be privately profitable. Significant criticism has come from Oren (1997) and Joskow and Tirole (2000), who argue that congestion contracts based on LMPs will inadvertently enhance the market power of certain generators. More recently, Joskow and Tirole (2005) have demonstrated that the economic efficiency of the market-based merchant transmission model falls apart when the underlying assumptions are relaxed. Blumsack (2006) discusses a network topology in which merchant investors can profit from modifying the network in ways that cause congestion, even without relaxing the assumptions of Bushnell and Stoft (1996, 1997).

The global experience with this "strong" version of market-based merchant transmission has been mixed. Littlechild and Skerk (2005a, 2005b) claim success in getting a small number of merchant lines built in Argentina. Australia's experience has been less successful; the merchant transmission lines built there have been sold back to the government due to lower-than-expected congestion revenues. The U.S. experience has involved a "weak" version of merchant transmission. Non-utility lines have been built, but compensation has come through long-term contracts and not through market prices. Not a single market-based merchant transmission project has been built in North America.

Underlying the market-based merchant transmission model is an implicit assumption that transmission projects can be cleanly separated into those that relieve congestion and those that enhance reliability. Since merchant transmission investments would be compensated with contracts based on congestion rents, this model would presumably only apply to those investments justifiable on "economic" grounds; investments for reliability would need to be socialized since the benefits are more widespread. This distinction has been made even more explicit in more recent "participant funding" model (Hebert 2004, Roark 2006).

Joskow (2005) argues that the distinction between investments for economics and those for reliability amounts to a meaningless dichotomy, since most transmission investments in the U.S. have been made by regulated utilities on the basis of reliability criteria. This paper and its companion piece (Blumsack, Ilić, and Lave 2006) demonstrate that this distinction is not simply meaningless; in many cases, it is wrong. Lines that cause congestion in the network may be justified on reliability grounds, and vice versa.

III. A STYLIZED EXAMPLE

The interaction between congestion and reliability can be illustrated using the simple four-bus test network shown in Figure 1. This test system is known as the Wheatstone network, and the link connecting buses 2 and 3 is known as the Wheatstone bridge. Bus 1 is assumed to have an inexpensive generator with $P_{G1}^{\max}$ = 100 MW, while bus 4 has an expensive generator and a load with a constant per-period real power demand of P_{L1} = 100 MW. Buses 2 and 3 are tie-points containing neither generation nor load. The susceptances of lines S_{12} and S_{34} are assumed to be equal, and the susceptances of lines S_{13} and S_{24} are assumed to be equal. The two "upstream" lines (S_{12} and S_{13}) have a rated limit of 55 MW each, and the two "downstream" lines (S_{24} and S_{34}) have a rated limit of 100 MW each.

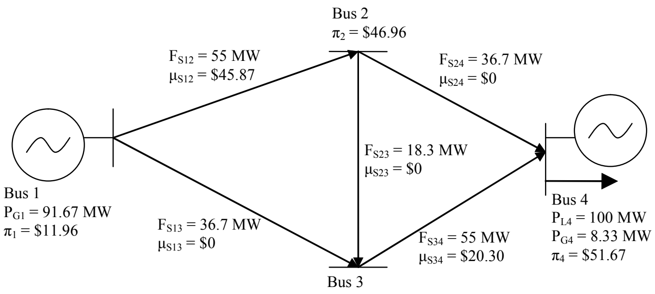

Figure 1: The addition of the Wheatstone bridge connecting buses 2 and 3 causes congestion along links S_{12} and S_{34}. The total system cost rises from $1,620 per hour without the Wheatstone bridge to $1,945 per hour with the bridge.

The cost curves of the two generators are given the following quadratic parameterization:

$$C(P_{G1}) = 200 + 10.3P_{G1} + 0.009P_{G1}^2 \tag{1}$$

$$C(P_{G4}) = 300 + 50P_{G4} + 0.1P_{G4}^2. \tag{2}$$

Consider a parallel version of the network in Figure 1, without the Wheatstone bridge. Abstracting from losses and reactive power demand, the inexpensive generator at bus 1 will produce 100 MW and serve the entire load. Due to the symmetry in the network, 50 MW will flow along each of the two paths from bus 1 to bus 4. A DC optimal power flow run on the parallel network yields identical LMPs of $12.11/MWh at each of the four buses; the total system cost of serving the load is $1,620 per period.

Once the Wheatstone bridge is added to the network, the pattern of flows shifts, causing congestion on lines S_{12} and S_{34} in the network.[1] The congestion reduces the total transfer capacity from bus 1 to bus 4. The generator at bus 1 produces only 91.67 MW of real power, while the remainder must be made up by the expensive generator at bus 4 (or load-shedding must occur at bus 4). The presence of congestion alters the LMPs at all four buses in the system; in particular, the LMP at the load bus increases to $51.67/MW-period. The total system cost of serving the load rises to $1,945 per period. For a system with market-based nodal pricing for generators and loads, the price paid by the load would increase by more than a factor of four, resulting in both wealth transfers from consumers to generators and a deadweight social loss (Blumsack and Ilić 2006, Joskow and Tirole 2005).

The phenomenon that congestion is caused or worsened by adding links to a network is known as the Braess Paradox, and

---

[1] These lines correspond to the high-susceptance lines in the network.

was first observed in automotive highway networks (Braess 1968, Arnott and Small 1994). Blumsack and Ilić (2006) provide a detailed discussion of the conditions under which the Braess Paradox will arise for more general electric power networks.

Thus, adding a Wheatstone bridge to a parallel system causes congestion and harms the network. However, the line might be justified on reliability grounds. Without the Wheatstone bridge, the network would violate the $N - 1$ criterion for transmission. Suppose an outage were to occur on either of the two "downstream" lines S_{24} or S_{34}. Without the Wheatstone bridge in the system, the loss of one of these downstream lines would effectively remove one of the electrical paths between buses 1 and 4. Thus, only 55 MW could be generated at bus 1. If the generator at bus 4 has $P_{G4}^{\max} < 45$ MW, then blackouts will result at bus 4.

IV. DECOMPOSITION OF SINGLE AND MULTIPLE ELEMENT CHANGES IN NETWORK TOPOLOGY

In the four-bus Wheatstone test network discussed in Section 3, addition of the Wheatstone bridge causes congestion, but adds to the reliability of the system in the event of an outage in one of the network boundary links. Through the example in Section 3, we have implicitly defined the congestion effect of a given line as being related to a single-element change in the network topology (namely, the addition of the Wheatstone bridge to the network). Similarly, we have defined the reliability effect as being associated with a multiple-element change in the network topology (the addition of the Wheatstone bridge and the loss of one of the transmission lines on the boundary). We will explicitly define congestion and reliability metrics in Section 5, but the simple example of the Wheatstone network makes clear that any thorough mathematical exploration of congestion and reliability requires us to decompose the effects of multiple changes to the network topology. To arrive at this decomposition, we adopt the DC power flow approximation and generalize the method of Ejebe and Wollenberg (1979), which models network outages as changes to the system admittance matrix.

The decomposition method proposed by Ejebe and Wollenberg, and expanded upon by Irissari, Levner, and Sasson (1979), was developed for the purpose of analyzing and ranking single-element contingencies. Thus, their modifications to the system admittance matrix (or just the susceptance matrix in the case of the DC power flow) amounted to network outages, modeled as $\Delta B_k = -B_k$ for an outage on the *k*th line. We generalize their calculations to include network additions, modeled as $\Delta B_k = +B_k$, and multiple-element topology changes.

A. *Decomposition of Single-Element Topology Changes*

We first review the case of a single-element topology change, as in Ejebe and Wollenberg (1979) and Irisarri, Levner, and Sasson (1979); an equivalent calculation was performed by Blumsack (2006, Ch. 3) for the case of the Wheatstone network presented in Section 3. The goal is to decompose the effect of a change in network topology so as to be able to write:

$$\mathbf{F}^{new} = \mathbf{F}^{old} + \Delta\mathbf{F}$$

where $\Delta\mathbf{F}$ represents the adjustment factor due to a single change in the network topology. The decomposition will show that $\Delta\mathbf{F}$ depends only on the system node-line adjacency matrix, the system susceptance matrix prior to the topology change, and the bus voltage angles prior to the topology change.

We start with the DC model for net injections:

$$\mathbf{P} = \mathbf{B}\boldsymbol{\theta}. \tag{4}$$

We model a change in the network topology as an adjustment to the susceptance matrix $\mathbf{B}$. The adjustment takes the form $\Delta\mathbf{B}_k = \mathbf{A}\Delta\mathbf{B}_k^{diag}\mathbf{A}'$, where $\Delta\mathbf{B}_k^{diag}$ is a diagonal matrix whose entries are all equal to zero except the *k,k*th entry, which is equal to ΔB_k (thus, ΔB_k represents a single-element change in the network topology affecting line *k*). In the case of a line outage, we will have $\Delta B_k = -B_k$, and the dimensionality of $\Delta\mathbf{B}_k^{diag}$ will be ($NL \times NL$). In the case of a network addition, we will have $\Delta B_k = +B_k$. The dimensionality of $\Delta\mathbf{B}_k^{diag}$ will be ($NL+1 \times NL+1$), and the dimensionality of $\mathbf{A}$ will be ($NB \times NL+1$), to account for the new line in the system.[2] Whether $\Delta\mathbf{B}_k^{diag}$ represents a line outage or a line addition in the network, we note that $\Delta\mathbf{B}_k^{diag}$ has rank one, and thus $\Delta\mathbf{B}_k$ also has rank one.

Following the change to the network topology, the DC equations can be written as:

$$\mathbf{P} = (\mathbf{B} + \Delta\mathbf{B}_k)\boldsymbol{\theta}. \tag{5}$$

Since $\Delta\mathbf{B}_k$ has rank one, we can write $\Delta\mathbf{B}_k = \Delta B_k \mathbf{A}_k \mathbf{A}'_k$, where $\mathbf{A}_k$ is the *k*th column of the node-line incidence matrix $\mathbf{A}$.

Solving for $\boldsymbol{\theta}$, and using the Sherman-Morrison-Woodbury matrix inversion lemma (Sherman and Morrison 1949, Woodbury 1950), we get:

$$\begin{aligned}\boldsymbol{\theta}^{new} &= (\mathbf{B} + \Delta\mathbf{B}_k)^{-1}\mathbf{P} \\ &= \left(\mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{A}_k(1 + \Delta B_k \mathbf{A}'_k\,\mathbf{B}^{-1}\mathbf{A}_k)^{-1} B_k \mathbf{A}'_k\,\mathbf{B}^{-1}\right)\mathbf{P} \\ \Rightarrow \boldsymbol{\theta}^{new} &= \boldsymbol{\theta}^{old} - \frac{\Delta B_k \mathbf{B}^{-1}\mathbf{A}_k \mathbf{A}'_k\,\boldsymbol{\theta}^{old}}{1 + \Delta B_k \mathbf{A}'_k\,\mathbf{B}^{-1}\mathbf{A}_k},\end{aligned} \tag{6}$$

---

[2] We could also model upgrades to the existing topology in a similar fashion. A line upgrade (not a line addition) would be represented by $\Delta B_k = +B_k$, with $+B_k$ being the magnitude of the upgrade. The dimensionality of $\Delta\mathbf{B}^{\text{diag}}$ would continue to be ($NL \times NL$).

where we note that $1 + B_k \mathbf{A}'_k \mathbf{B}^{-1} \mathbf{A}_k$ is a scalar. We can rewrite this as:

$$\boldsymbol{\theta}^{new} \quad = \boldsymbol{\theta}^{old} - \gamma \mathbf{r}\,, \tag{7}$$

where $\mathbf{r} = \mathbf{B}^{-1}\mathbf{A}_k$ and $\gamma = \dfrac{\Delta B_k \mathbf{A}'_k\, \boldsymbol{\theta}^{old}}{1 + \Delta B_k \mathbf{A}'_k\, \mathbf{r}}$. In the discussion below, it will be easier to define $\Delta\boldsymbol{\theta} = \boldsymbol{\theta}^{new} - \boldsymbol{\theta}^{old}$. Note that γ is a scalar and **r** is a vector of dimension (NB × 1). Plugging (7) into the DC flow equations yields:

$$\begin{aligned} \mathbf{F}^{new} &= \mathbf{B}^{diag}\mathbf{A}'(\boldsymbol{\theta}^{old} - \gamma \mathbf{r}) \\ &= \mathbf{F}^{old} - \gamma \mathbf{B}^{diag}\mathbf{A}'\mathbf{r}. \end{aligned} \tag{8}$$

The equivalent of (8) for flow on a single line F_l, is (Irisarri, Levner, and Sasson 1979, Blumsack 2006 Ch. 3):

$$F_l^{new} = \begin{cases} F_l^{old} + b_k^{-1}\mathbf{A}'_l\,\mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{old}, & l \neq k \\ \left(F_l^{old} - \Delta B_l \delta_l^{old}\right)(1 - b_l^{-1}\mathbf{A}'_l\,\mathbf{B}^{-1}\mathbf{A}_l), & l = k, \end{cases}$$

(9)

where $b_k = (\Delta B_k^{-1} + \mathbf{A}_k'\mathbf{B}^{-1}\mathbf{A}_k)$, $b_l = (\Delta B_l^{-1} + \mathbf{A}_l'\mathbf{B}^{-1}\mathbf{A}_l)$.

B. Decomposition of Multiple-Element Topology Changes

We now consider the case of a change in the network topology affecting n distinct network elements. The network adjustment is again modeled as an adjustment of the form $\Delta\mathbf{B} = \mathbf{A}\Delta\mathbf{B}^{diag}\mathbf{A}'$, except we allow $\Delta\mathbf{B}^{diag}$ to have multiple non-zero entries ΔB_{kk}, $k = \{1,\ldots,n\}$ on the main diagonal (the off-diagonal entries are still assumed to be zero). If we assume that $n1$ of these network changes represent line outages, and $n2$ of these network changes represent new lines (so we have $n1 + n2 = n$), then the dimensionality of $\Delta\mathbf{B}^{diag}$ will be ($NL + n2$ × $NL + n2$). For example, we would model the addition of line k and an outage on line $m \neq k$ with the ($NL+1$ × $NL+1$) matrix:

$$\Delta\mathbf{B}^{diag} = \begin{bmatrix} 0 & & \cdots & & 0 \\ & \ddots & & & \\ & & +B_k & & \\ \vdots & & & \ddots & & \vdots \\ & & & & -B_m & \\ & & & & & \ddots \\ 0 & & \cdots & & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \cdots & 0 \\ & \ddots & \\ \vdots & +B_k & \vdots \\ & \ddots & \\ 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\ & \ddots & \\ \vdots & -B_m & \vdots \\ & & \ddots \\ 0 & \cdots & 0 \end{bmatrix}$$

$$= \Delta\mathbf{B}_k^{diag} + \Delta\mathbf{B}_m^{diag}.$$

More generally, we can write an n-element network topology change as:

$$\Delta\mathbf{B}^{diag} = \Delta\mathbf{B}_1^{diag} + \Delta\mathbf{B}_2^{diag} + \cdots + \Delta\mathbf{B}_n^{diag}\,. \tag{10}$$

We note that the rank of $\Delta\mathbf{B}^{diag}$ is equal to n, the number of distinct adjustments to the network topology. Thus, the rank of $\Delta\mathbf{B} = \mathbf{A}\Delta\mathbf{B}^{diag}\mathbf{A}'$ is also equal to n.

The modified DC model for a multi-element topology change can be written as:

$$\mathbf{P} = (\mathbf{B} + \mathbf{A}\Delta\mathbf{B}^{diag}\mathbf{A}')\boldsymbol{\theta}\,. \tag{11}$$

Solving for the vector of bus voltage angles, we get:

$$\boldsymbol{\theta} = (\mathbf{B} + \mathbf{A}\Delta\mathbf{B}^{diag}\mathbf{A}')^{-1}\mathbf{P}\,. \tag{12}$$

Invoking the matrix inversion lemma, this can be written as:

$$\boldsymbol{\theta} = \left(\mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{A}(\mathbf{I} + \Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}\right)\mathbf{P}\,, \tag{13}$$

where **I** is the (NL × NL) identity matrix. The adjustment factor for the vector of bus angles becomes:

$$\Delta\boldsymbol{\theta} = -\mathbf{B}^{-1}\mathbf{A}(\mathbf{I} + \Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\Delta\mathbf{B}^{diag}\mathbf{A}'\boldsymbol{\theta}^{old}. \tag{14}$$

If we let $\mathbf{R}=\mathbf{B}^{-1}\mathbf{A}$, and $\mathbf{C} = (\mathbf{I} + \Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}\mathbf{A})$, we get:

$$\Delta\boldsymbol{\theta} = -\mathbf{R}\mathbf{C}^{-1}\mathbf{A}'\boldsymbol{\theta}^{old}. \tag{15}$$

Based on O'Leary and Widlund (1979) and Agkün et. al. (2001), we note that **C** is invertible if and only if $\mathbf{B} + \Delta\mathbf{B}^{diag}\mathbf{A}\mathbf{A}'$ is invertible. If we let $\boldsymbol{\Gamma} = \mathbf{C}^{-1}\mathbf{A}'\boldsymbol{\theta}^{old}$, then we can write:

$$\Delta\boldsymbol{\theta} = -\mathbf{R}\boldsymbol{\Gamma} = -\gamma_1 \mathbf{r}_1 - \gamma_2 \mathbf{r}_2 - \cdots - \gamma_{NL} \mathbf{r}_{NL} \,, \tag{16}$$

where γ_i is the ith element of $\boldsymbol{\Gamma}$, and $\mathbf{r}_i$ is the ith column of $\mathbf{R}$.

Thus, the voltage-angle adjustment factor for an n-element topology change is a linear combination of n single-element topology changes. As with the single-element analysis in (8), we can use the voltage-angle adjustment factor $\Delta\boldsymbol{\theta}$ in the DC flow equations to calculate the effect of a multi-element topology change on the network flows, as follows:

$$\begin{aligned}\mathbf{F}^{new} &= \left(\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag}\right)\mathbf{A}'(\boldsymbol{\theta}^{old} - \mathbf{R}\boldsymbol{\Gamma}) \\ &= \mathbf{F}^{old} - \left(\Delta\mathbf{B}^{diag}\boldsymbol{\delta}^{old} - \sum_{i=1}^{NL}\gamma_i\left(\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag}\right)\mathbf{A}'\mathbf{r}_i\right).\end{aligned} \tag{17}$$

Similarly, the effect on the net nodal injections can be calculated as:

$$\begin{aligned}\mathbf{P}^{new} &= \mathbf{A}\left(\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag}\right)\mathbf{A}'(\boldsymbol{\theta}^{old} - \mathbf{R}\boldsymbol{\Gamma}) \\ &= \mathbf{P}^{old} - \left(\mathbf{A}\Delta\mathbf{B}^{diag}\boldsymbol{\delta}^{old} - \sum_{i=1}^{NL}\gamma_i\mathbf{A}\left(\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag}\right)\mathbf{A}'\mathbf{r}_i\right).\end{aligned} \tag{18}$$

Equations (17) and (18) give us a way to decompose the effects of multiple changes to the network topology. In particular, we see that the adjustment to the vector of network flows or injections due to a multi-element change in the network topology is a linear combination of the effects of a series of single-element changes to the network topology.

V. VARIATION OF DISTRIBUTION FACTORS WITH TOPOLOGICAL CHANGES

We can use the same method developed in Section 4 to derive an expression showing how the matrix of power transfer distribution factors (PTDF) changes following alterations to the network topology. We begin with the definition of the PTDF matrix:

$$\boldsymbol{\rho} = \mathbf{H}\mathbf{B}^{-1} \,, \tag{19}$$

Where $\mathbf{H}$ is a ($NL \times NB$) matrix defined by $\mathbf{H} = \mathbf{B}^{diag}\mathbf{A}'$. Thus, we can rewrite (19) as:

$$\boldsymbol{\rho} = \mathbf{B}^{diag}\mathbf{A}'(\mathbf{A}\mathbf{B}^{diag}\mathbf{A}')^{-1} \,. \tag{19'}$$

Following a change to the network topology represented by $\Delta\mathbf{B}^{diag}$, we have:

$$\boldsymbol{\rho}^{new} = (\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag})\mathbf{A}'\left[\mathbf{A}(\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag})\mathbf{A}'\right]^{-1} . \tag{20}$$

Distributing terms in the inverse matrix in (20), we get:

$$\boldsymbol{\rho}^{new} = (\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag})\mathbf{A}'\left[\mathbf{B} + \mathbf{A}\Delta\mathbf{B}^{diag}\mathbf{A}'\right]^{-1} . \tag{20'}$$

If we define $\Delta\mathbf{B} = -\mathbf{B}^{-1}\mathbf{A}(\mathbf{I} + \Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}$, as in (13), then we can rewrite (20) as:

$$\begin{aligned}\boldsymbol{\rho}^{new} &= (\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag})\mathbf{A}'\left[\mathbf{B}^{-1} + \Delta\mathbf{B}^{-1}\right] \\ &= \left(\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1}\right) + \mathbf{B}^{diag}\mathbf{A}'\Delta\mathbf{B}^{-1} + \Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1} \\ &\qquad + \Delta\mathbf{B}^{diag}\mathbf{A}'\Delta\mathbf{B}^{-1} \\ &= \boldsymbol{\rho}^{old} + \Delta\boldsymbol{\rho},\end{aligned} \tag{20''}$$

where $\Delta\boldsymbol{\rho} = \mathbf{B}^{diag}\mathbf{A}'\Delta\mathbf{B}^{-1} + \Delta\mathbf{B}^{diag}\mathbf{A}'\mathbf{B}^{-1} + \Delta\mathbf{B}^{diag}\mathbf{A}'\Delta\mathbf{B}^{-1}$.

Equation (20'') describes the change in the distribution factors for a general (multi-element) topological network change. The equivalent of (20'') for the effect of a single-element topological change ΔB_k on the lth line is given by:

$$\boldsymbol{\rho}_l^{new} = \boldsymbol{\rho}_l^{old} + B_l\mathbf{A}'_l\,\Delta\mathbf{B}^{-1} + \Delta B_k(\mathbf{A}'_l\,\Delta\mathbf{B}^{-1} + A'_l\,\mathbf{B}^{-1}) \,, \tag{21}$$

where $\boldsymbol{\rho}_l$ is the lth row of the distribution matrix, and $\mathbf{A}_l$ is the lth row of the node-line adjacency matrix.

We can use the DC flow model $\mathbf{F}=\boldsymbol{\rho}\mathbf{P}$ to link the distribution-matrix decomposition in (21) with the flow-vector decomposition (17). We note that:

$$\begin{aligned}\mathbf{F}^{new} &= (\boldsymbol{\rho}^{old} + \Delta\boldsymbol{\rho})\mathbf{P} \\ &= \boldsymbol{\rho}^{old}\mathbf{P} + \Delta\boldsymbol{\rho}\mathbf{P} \\ &= \mathbf{F}^{old} + \Delta\boldsymbol{\rho}\mathbf{P}.\end{aligned} \tag{22}$$

Since $\mathbf{F}^{new} = \mathbf{F}^{old} + \Delta\mathbf{F}$, it must be true that $\Delta\mathbf{F} = \Delta\boldsymbol{\rho}\mathbf{P}$, or equivalently:

$$\Delta\boldsymbol{\rho}\mathbf{P} = -\left[\Delta\mathbf{B}^{diag}\boldsymbol{\delta}^{old} - \left(\mathbf{B}^{diag} + \Delta\mathbf{B}^{diag}\right)\mathbf{A}'\mathbf{R}\boldsymbol{\Gamma}\right]. \tag{23}$$

Thus, the effects of a n-element topology change on the PTDF matrix can be broken down into a linear combination of n single-element topology changes. Equations (23) and (17) can essentially be used interchangeably.

VI. CONGESTION AND RELIABILITY METRICS

The example of the Wheatstone network in Section 3 suggests two conceptual distinctions between reliability and congestion. The first is that reliability refers to the state or robustness of the network under contingencies, while congestion is a system attribute associated with normal operations. In developed countries with robust power grids, contingencies should not occur very often, but particular paths may become congested on a regular basis, particularly during times of peak demand. Thus, congestion events may be common and perhaps even predictable. Reliability problems (which lead to demand not being fully served), on the other hand, are the result of contingencies that should be random and rare.

The second distinction is that the presence of congestion may increase the cost of filling customer demands, while a lack of reliability in the system results in the physical inability of the system to meet these demands. Here, we are explicitly defining both reliability and congestion from the point of view of the customer. If a particular piece of equipment in the network is prone to outages, but these outages do not restrict the amount of customer demand that the network can meet, then the network is sufficiently reliable for the customer.

Before discussing the extent to which these two system attributes can be decomposed, we define some explicit congestion and reliability metrics. Our focus in this paper is the effects of topological changes on both reliability and congestion in the network. For concreteness, we will define a network topology Ω as a triple $\Omega = \{\mathbf{B}, \mathbf{A}, \mathbf{F}^{\max}\}$, where **B** is the susceptance matrix, **A** is the node-line adjacency matrix, and $\mathbf{F}^{\max}$ is the vector of capacity limits for the transmission lines. Thus, there is a distinction between our definition of a topology and the usual definition of a graph or network as a collection of nodes and edges.

Unless stated otherwise, we will be comparing different network topologies under the following set of assumptions:

Assumption 1: The profile of desired nodal demands $\mathbf{P}_d = (P_{d1}, \ldots, P_{dNB})$ does not change with a change in network topology.[3]

Assumption 2: Demand is completely price-inelastic.

Assumption 3: The grid operator treats all customers as if they valued reliability equally. One important implication of this assumption is that unserved energy at a given node is inconsistent with excess generation capacity at that node.[4]

A. Congestion Metrics

Congestion in an electric network occurs whenever a transmission line is loaded up to its predetermined capacity limit. Actual transmission lines have a variety of constraints representing different operating states of the system. These include voltage stability loading limits, thermal limits, and short-term contingency limits. These are normally measured in MVA to accommodate both real and reactive power constraints, but to simplify the discussion we will treat transmission lines as if they had a single steady-state capacity limit, stated in MW for real power.

A single line *l* becomes congested when $F_l = F_l^{\max}$. From (9), we can see that a single-element topology change affecting line $k \neq l$ will cause congestion if $F_l^{new} \geq F_l^{\max}$, or:

$$\begin{aligned} & F_l^{old} + b_k^{-1}\mathbf{A}'_l\,\mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{old} \geq F_l^{\max} \\ & \Rightarrow \Delta B_k^{-1} \geq \frac{\mathbf{A}'_l\,\mathbf{B}^{-1}\mathbf{A}_k B_l \delta_k^{old}}{F_l^{\max} - F_l^{old}} - \mathbf{A}_k'\mathbf{B}^{-1}\mathbf{A}_k . \end{aligned} \tag{24}$$

We can also express (24) using the network distribution matrix, as in Section 5. An equivalent condition to (24) is:

$$\Delta B_k > \frac{F_l^{\max} - F_l^{old} - B_l(\mathbf{A}'_l\,\Delta\mathbf{B}^{-1}\mathbf{B}\boldsymbol{\theta})}{\delta_l^{old} + \mathbf{A}'_l\,\Delta\mathbf{B}^{-1}\mathbf{B}\boldsymbol{\theta}} . \tag{24'}$$

For a given demand profile in the network, (24) can serve as the definition of congestion caused by a specific network link. That is, for a given vector of nodal demands, we define line *k* as causing congestion on line *l* (relative to the network topology without line *k*) if (24) is satisfied.

Congestion may impose a cost on the system if it results in generating units being dispatched out of merit order. We measure the congestion cost *CC* of a single-element or multi-element change to the network topology by taking the difference in the total cost of serving the load before and after the topological change:

$$CC = \sum_i C_i\left(P_{Gi,opt}^{\Omega_a}\right) - \sum_i C_i\left(P_{Gi,opt}^{\Omega_b}\right), \tag{25}$$

where Ω_a and Ω_b represent two distinct network topologies, and $P_{Gi,opt}^{\Omega_a}, P_{Gi,opt}^{\Omega_b}$ represent the optimal output of the *i*th generating unit with respect to network topologies Ω_a and Ω_b.[5]

Equation (24) provides an explicit condition under which a topological change will result in network congestion (for a given network demand profile). Measuring the congestion cost requires optimization of the entire system under different network topologies. Thus, the magnitude of the network congestion cost is largely an empirical matter, rather than a theoretical one.

B. Reliability Metrics

Current industry practice assesses network reliability using a number of different metrics, such as:

1. The $N - k$ criterion; whether the system can continue to provide uninterrupted service to customers in the

---

[3] There is some foreshadowing in this assumption. We will define reliability (from the point of view of the customer) as the situation where there is no difference between the actual demand profile and the profile of desired demand.

[4] Another implication is that without some cost-based or value-based decision criteria, it may not always be clear which customer or customers should be blacked out in the event of a physical shortage of network transmission resources. We abstract from this decision problem and focus on the end state of the network; i.e., the sufficient system conditions for a customer to be blacked out.

[5] By "optimal output," we mean the level of output resulting from an optimization problem which minimizes the total cost of generation.